

Strategic Decompositions of Normal Form Games: Zero-sum Games and Potential Games

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Abstract

We study new classes of games, called *zero-sum equivalent games* and *zero-sum equivalent potential games*, and prove decomposition theorems involving these classes of games. We say that two games are “strategically equivalent” if, for every player, the payoff differences between two strategies (holding other players’ strategies fixed) are identical. A zero-sum equivalent game is a game that is strategically equivalent to a zero-sum game; a zero-sum equivalent potential game is a zero-sum equivalent game that is strategically equivalent to a common interest game. We also call a game “normalized” if the sum of one player’s payoffs, given the other players’ strategies, is always zero. We show that any normal form game can be uniquely decomposed into either (i) a zero-sum equivalent game and a normalized common interest game, or (ii) a zero-sum equivalent potential game, a normalized zero-sum game, and a normalized common interest game, each with distinctive equilibrium properties. For example, we show that two-player zero-sum equivalent games with finite strategy sets generically have a unique Nash equilibrium and that two-player zero-sum equivalent potential games with finite strategy sets generically have a strictly dominant Nash equilibrium.

Keywords: decomposition, zero-sum games, potential games.

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1. Introduction

When two people start a joint venture, their interests are aligned. In the division of a pie or unclaimed surpluses, however, someone's gain always comes at the expense of somebody else. So-called common interest games and zero-sum games serve as polar models for studying these social interactions. Two games can be regarded as “strategically equivalent” if, for every player, the payoff differences between two strategies (holding other players' strategies fixed) are identical. That is, in two strategically equivalent games, strategic variables such as best responses of players are identical.¹ Potential games—a much studied class of games in the literature—are precisely those games that are strategically equivalent to common interest games. We also call a game “normalized” when the sum of one player's payoffs, given the other players' strategies, is always zero.

We are interested in zero-sum games and their variants—(i) games that are strategically equivalent to a zero-sum game, accordingly named *zero-sum equivalent games*, (ii) zero-sum equivalent games which are, at the same time, equivalent to a common interest game, named *zero-sum equivalent potential games* and (iii) *normalized zero-sum games*. Our interest in zero-sum equivalent games is motivated by their analogous definitions to potential games. Potential games retain all the attractive properties of common interest games (the existence of a pure strategy Nash equilibrium and a potential function) because they are strategically equivalent to common interest games. Thus, zero-sum equivalent games are expected to retain similar desirable properties of zero-sum games as well. It is well-known that two-player zero-sum games with a finite number of strategies have mini-max strategies and admit a value of games which are useful tools for amenable analysis of equilibrium. Recently, there have been generalizations of these properties and characterizations for a special class of n -player zero-sum games (Bregman and Fokin (1998); Cai et al. (2015); see Section 3 for an extensive discussion of zero-sum equivalent games).

To study these classes of zero-sum related games, we develop decomposition methods of an arbitrary normal form game and obtain several constituent components belonging to these classes and a special class of common interest games. Our decomposition results provide (i) characterizations for zero-sum equivalent games as

¹See Definition 2.2 and Monderer and Shapley (1996) and Weibull (1995); see also Morris and Ui (2004) for best response equivalence.

4,4	-1,1	1,-1
1,-1	2,2	-2,0
-1,1	0,-2	2,2

$$=$$

2,2	-1,-1	-1,-1
-1,-1	2,2	-1,-1
-1,-1	-1,-1	2,2

$$+$$

0,0	-1,1	1,-1
1,-1	0,0	-1,1
-1,1	1,-1	0,0

$$+$$

1,1	1,0	1,0
0,1	0,0	0,0
0,1	0,0	0,0

$$+$$

1,1	0,1	0,1
1,0	0,0	0,0
1,0	0,0	0,0

$\underbrace{\hspace{10em}}_{=C}$

$\underbrace{\hspace{10em}}_{=Z}$

$\underbrace{\hspace{10em}}_{=B}$

$\underbrace{\hspace{10em}}_{=E}$

Table 1: **Illustration of our four-component decomposition.** This example illustrates our main results. Here, C is a common interest game, Z is a zero-sum game (the Rock-Paper-Scissors game), B is a game in which the first strategy is the dominant strategy, and E is called a “non-strategic” game, in which, for every player, the payoff differences between two strategies (holding other players’ strategies fixed) are identical (see Definition 2.1). Observe that C and Z are “normalized” in the sense that the column sums and row sums of the payoffs are zeros.

well as others (e.g., Propositions 3.2, 4.1, and 4.2) and (ii) decompositions of a given game into components with distinctive equilibrium properties (Theorem 5.1, Figure 3). Intuitively, a decomposition of an arbitrary game gives an idea of how the original game is “close” to component games with desirable properties.

Our main results (Theorem 2.1 (ii) and Proposition 4.1) show that every normal form game can be decomposed into four components: (i) a “normalized” common interest component (C in Table 1), (ii) a “normalized” zero-sum component (Z in Table 1), (iii) a zero-sum equivalent potential component—component equivalent to both a zero-sum and common interest game (B in Table 1), and (iv) a nonstrategic component (E in Table 1). Most popular zero-sum games, such as Rock-Paper-Scissors games and Matching Pennies games, belong to the class of normalized zero-sum games (see also Cyclic games in Hofbauer and Schlag (2000)).

This study makes the following contributions. We develop a more general way of decomposing normal form games than existing methods. Existing decomposition methods of normal form games, such as in Hwang and Rey-Bellet (2011), Candogan et al. (2011), and Sandholm (2010a), are limited to finite strategy set games, relying on decomposition methods of tensors (or matrices) or graphs. Our new insights lie in viewing the set of all games as a Hilbert space; we use a Hilbert space decomposition technique and obtain decomposition results of normal form games at an abstract level, which are applicable to games with continuous strategy

sets as well as those with finite strategy sets.² In this way, our method shows a unified and transparent mechanism of decompositions of games and can be easily modified to decompose other classes of games such as Bayesian games. Further, many problems in economics are modeled using continuous strategy games; thus our results, such as the characterizations and the equilibrium properties of component games, can be applied to a wide range of problems including, for example, Cournot games and contest games.

Second, the concept of orthogonality in Hilbert space is useful in the sense that we can naturally characterize a class of games by describing the games which are orthogonal to it, that is their *orthogonal complements*. For example, the sufficiency and necessity of the well-known Monderer and Shapley cycle condition for potential games (Theorem 2.8 in [Monderer and Shapley \(1996\)](#)) can be proved by showing that this condition requires that potential games are orthogonal to all the normalized zero-sum games. We use the orthogonality which yields a unique decomposition of a given game into a common interest game and a zero-sum game ([Kalai and Kalai, 2010](#)) and into a normalized game and a non-strategic game ([Hwang and Rey-Bellet, 2011](#); [Candogan et al., 2011](#)). Based on this orthogonal decomposition, we are able to provide characterizations of games, such as zero-sum equivalent games, zero-sum potential games, normalized zero-sum games, and normalized common interest games.

After our main decompositions in Section 2, we present detailed analyses of each component game—zero-sum equivalent games, zero-sum equivalent potential games, normalized zero-sum games, and normalized common interest games. In Section 3, we show that zero-sum equivalent games have a Nash equilibrium and under some (easily checkable) conditions they have a unique Nash equilibrium. We also show that show that two-player zero-sum equivalent games with finite strategy sets generically have a unique Nash equilibrium (Corollary 3.1). In Section 4, we study zero-sum equivalent potential games. In particular, we show that two-player games in this class generically possess a strictly dominant strategy (Corollary 4.3). This kind of two-player game (game B in Table 1), which is strategically equivalent to both a zero-sum game and a common interest game, only involves self-interactions, not

²For example, the decomposition by [Candogan et al. \(2011\)](#) relies on the Helmholtz decomposition of flows on the graph for games with finite strategy sets and uses Moore inverses of matrix operators. In fact, one of our previous decomposition results in [Hwang and Rey-Bellet \(2011\)](#) have some overlaps with the main result in the cited paper.

Notations	Names	Definitions
\mathcal{C}	Common interest games	Definition 2.1
\mathcal{Z}	Zero-sum games	Definition 2.1
\mathcal{N}	Normalized games	Definition 2.1
\mathcal{E}	non-strategic games (define E quivalence relation)	Definition 2.1
$\mathcal{Z} + \mathcal{E}$	Zero-sum E quivalent games	Definition 2.4
$\mathcal{C} + \mathcal{E}$	potential games (C ommon interest E quivalent games)	Definition 2.3
$\mathcal{B} = (\mathcal{Z} + \mathcal{E}) \cap (\mathcal{C} + \mathcal{E})$	(B oth) zero-sum equivalent (and) potential games	Definition 2.4
\mathcal{D}	games which have a D ominant strategy for two-player games	Definition 4.1
$\mathcal{N} \cap \mathcal{Z}$	Normalized Z ero-sum games	
$\mathcal{N} \cap \mathcal{C}$	Normalized C ommon interest games	

Table 2: **Notations for the spaces of games.**

interactions between two players, since a player’s payoffs depend only on her own actions.

Putting all this together, we provide a decomposition of a two player finite game into component games, each possessing distinctive equilibrium characteristics (Theorem 5.1). To help readers digest the main results, we keep some important proofs in the text; however, to streamline the presentation, we relegate some lengthy and tedious proofs to the appendix. Also, in Appendix A, we review the basic results about the decomposition of a Hilbert space to provide theoretical background.

2. Decomposition Theorems

In this section, we present the basic setup, provide some preliminary results and present our main decomposition theorems.

2.1. Basic setup

An n -player game (N, S, f) is specified by a set of players, $N = \{1, 2, \dots, n\}$; a set of strategy profiles, $S := \prod_{i=1}^n S_i$, where S_i is a set of strategies for player i ; and a payoff function, $f := (f^{(1)}, f^{(2)}, \dots, f^{(n)})$. For each i , $f^{(i)} : S \rightarrow \mathbb{R}$ is a real-valued function, with $f^{(i)}(s)$ specifying the payoff for player i , given a strategy profile, $s = (s_1, \dots, s_n)$. We follow the convention of writing $f^{(i)}(s_i, s_{-i}) = f^{(i)}(s)$, where s_{-i} is the strategy profile of the players other than i . We associate a finite measure space with each set of strategies, S_i , as follows. If S_i is a finite set, we let

m_i be the counting measure with the natural σ -algebra of all subsets of S_i . If S_i is a subset of \mathbb{R}^{d_i} for some d_i , we assume that S_i is bounded and choose m_i to be the Lebesgue measure on S_i with the Borel σ -algebra on S_i . Thus, we have $m_i(S_i) < \infty$. We call a game with finite strategy sets simply a finite game to distinguish it from continuous strategy games.

Given N and S , a game is uniquely specified by the vector-valued function $f := (f^{(1)}, f^{(2)}, \dots, f^{(n)})$ which we assume to be measurable. We succinctly write an n -fold integration as follows:

$$\int f^{(i)} dm = \int \cdots \int f^{(i)}(s_1, \dots, s_n) dm_1(s_1) \cdots dm_n(s_n).$$

We define the norm of the payoff function f by

$$\|f\| := \left(\sum_{i=1}^n \int |f^{(i)}|^2 dm \right)^{1/2}$$

and consider the space of games to be

$$\mathcal{L} := \{f : S \rightarrow \mathbb{R}^n \text{ measurable and } \|f\| < \infty\}. \quad (1)$$

For finite games, the payoff functions can be represented as matrices (or tensors), and \mathcal{L} is the set of all matrices (or tensors) of suitable dimensions. Existing decomposition methods, such as those of [Hwang and Rey-Bellet \(2011\)](#), [Candogan et al. \(2011\)](#) and [Sandholm \(2010a\)](#), all focus on the case of finite strategy sets, while we consider a general strategy space which can be either finite or continuous. We also follow the usual convention of redefining \mathcal{L} to be the set of all equivalent classes of almost everywhere defined integrable functions on S . Thus, whenever we state a result for a function, f , it holds for all its equivalent functions which agree with f almost everywhere. Our choice of the norm turns \mathcal{L} into a Hilbert space with a scalar product

$$\langle f, g \rangle := \sum_i \int f^{(i)} g^{(i)} dm, \quad (2)$$

which gives us $\langle f, f \rangle = \|f\|^2$. We next introduce several classes of games of interest.

Definition 2.1. We define the following subspaces of \mathcal{L} :

- (i) The space of *common interest games*, \mathcal{C} , is defined by

$$\mathcal{C} := \{f \in \mathcal{L} : f^{(i)}(s) = f^{(j)}(s) \text{ for all } i, j \text{ and for all } s\}.$$

(ii) The space of *zero-sum games*, \mathcal{Z} , is defined by

$$\mathcal{Z} := \{f \in \mathcal{L} : \sum_{l=1}^n f^{(l)}(s) = 0 \text{ for all } s\}.$$

(iii) The space of *normalized games*, \mathcal{N} , is defined by

$$\mathcal{N} := \{f \in \mathcal{L} : \int f^{(i)}(t_i, s_{-i}) dm_i(t_i) = 0 \text{ for all } s_{-i}, \text{ for all } i\}. \quad (3)$$

(iv) The space of *nonstrategic games*, \mathcal{E} , is defined by

$$\mathcal{E} := \{h \in \mathcal{L} : h^{(i)}(s_i, s_{-i}) = h^{(i)}(s'_i, s_{-i}) \text{ for all } s_i, s'_i, s_{-i}, \text{ for all } i\}. \quad (4)$$

Common interest games in \mathcal{C} and zero-sum games in \mathcal{Z} are familiar games, modeling cooperative and competitive interactions, respectively. Tractable equilibrium analysis is possible in both classes of games via potential functions and mini-max solutions.

To exhibit an important property of the normalized games in \mathcal{N} , let σ_i denote the uniform mixed strategy for player i ; i.e., the probability measure

$$d\sigma_i(s_i) = \frac{1}{m(S_i)} dm_i(s_i). \quad (5)$$

For normalized games in \mathcal{N} , for any player, the payoff for the uniform mixed strategy is zero against any strategy profile of her opponents. Because of this property, normalized common interest games and normalized zero-sum games always have the uniform mixed strategy as a Nash equilibrium (see Proposition 5.1). The term, “normalized”, stems from this game’s being obtained by removing a “non-strategic” component of a game—this will be explained in more detail in subsection 2.3.

By definition, for a non-strategic game in \mathcal{E} , (also sometimes called a passive game), the players’ payoffs do not depend on their own strategy choices (Sandholm (2010a); Candogan et al. (2011)). Thus, each player’s strategy choice plays no role in determining her payoff; any arbitrary strategy profile is a Nash equilibrium. Because of this property, the players’ strategic relations remain unchanged if we add the payoff of a non-strategic game to that of another game. This leads us to the definition of

strategic equivalence (and thus notation, \mathcal{E}).³

Definition 2.2. We say that game g is *strategically equivalent* to game f if

$$g = f + h \text{ for some } h \in \mathcal{E}$$

We write this relation as $g \sim f$, and this is obviously an equivalence relation.

The class of potential games defined by [Monderer and Shapley \(1996\)](#) has received much attention for its analytical convenience. Among the several equivalent definitions for potential games we choose the one convenient for our purpose: potential games are those that are strategically equivalent to common interest games.

Definition 2.3. The space of *potential games* (common interest equivalent games) is defined by

$$\mathcal{C} + \mathcal{E}. \tag{6}$$

Recall that the notation $f \in \mathcal{C} + \mathcal{E}$ means that f can be written as a linear combination of a common interest game and a non-strategic game.

It is also well-known that two-player zero-sum games (as well as potential games) have desirable properties—the value of a game, the mini-max theorem and so on. We then expect that (i) two-player zero-sum equivalent games retain some of these properties and (ii) n -person zero-sum equivalent games may possess similar but possibly weakened properties. As the example in [Section 2.2](#) shows, n -player zero-sum equivalent games have appealing properties, which can be useful for equilibrium analysis (see [Section 3](#)). This leads us to consider the space of games that is strategically equivalent to zero-sum games. Moreover, as both potential games and zero-sum equivalent games have desirable properties, we will identify, as well, the class of games which is simultaneously strategically equivalent to a common interest game and a zero-sum game.

Definition 2.4. We have the following definitions:

³One can study different strategic equivalences. For example, [Monderer and Shapley \(1996\)](#) introduce the concept of w -potential games in which the payoff changes are proportional for each player. [Morris and Ui \(2004\)](#) also study the best response equivalence of games in which players have the same best-responses. We choose our definition of strategic equivalence since it is most natural with the linear structure of the space of all games.

(i) The space of *zero-sum equivalent games* is defined by

$$\mathcal{Z} + \mathcal{E} \tag{7}$$

(ii) The space of games that is strategically equivalent to both a common interest game and a zero-sum game, called *zero-sum equivalent potential games*, is denoted by \mathcal{B} .

2.2. Motivating examples

Example 1 (Strategic equivalence: Cournot oligopoly).

Consider a quasi-Cournot oligopoly game with a linear demand function for which the payoff function for the i -th player, for $i = 1, \dots, n$, is given by

$$f^{(i)}(s) = (\alpha - \beta \sum_{j=1}^n s_j) s_i - c_i(s_i),$$

where $\alpha, \beta > 0$, $c_i(s_i) \geq 0$ for all $s_i \in [0, \bar{s}]$ for all i and for some sufficiently large \bar{s} .⁴ It is well-known that this game is a potential game (Monderer and Shapley 1996); i.e., it is strategically equivalent to a common interest game but is also strategically equivalent to a zero sum game (if $n \geq 3$). To show this, when $n = 3$, we write the payoff function as

$$\begin{pmatrix} f^{(1)} \\ f^{(2)} \\ f^{(3)} \end{pmatrix} = \underbrace{\begin{pmatrix} (\alpha - \beta s_1) s_1 - c_1(s_1) \\ (\alpha - \beta s_2) s_2 - c_2(s_2) \\ (\alpha - \beta s_3) s_3 - c_3(s_3) \end{pmatrix}}_{\text{Self-Interaction}} - \underbrace{\begin{pmatrix} \beta s_1 s_2 \\ \beta s_1 s_2 \\ 0 \end{pmatrix}}_{\text{Interactions between players 1 and 2}} - \underbrace{\begin{pmatrix} \beta s_1 s_3 \\ 0 \\ \beta s_1 s_3 \end{pmatrix}}_{\text{Interactions between players 1 and 3}} - \underbrace{\begin{pmatrix} 0 \\ \beta s_2 s_3 \\ \beta s_2 s_3 \end{pmatrix}}_{\text{Interactions between players 2 and 3}} \tag{8}$$

The self-interaction term is strategically equivalent to both a common interest

⁴Here, quasi-Cournot games allow the negativity of the price (Monderer and Shapley, 1996). Further, we can choose \bar{s} to ensure that the unique Nash equilibrium lies in the interior $[0, \bar{s}]$ as follows. Suppose that $c_i(s_i)$ is linear; that is, $c_i(s_i) = c_i s_i$ for all i . We assume that $\alpha > n \min_i c_i - (n-1) \max_i c_i$, which ensures that the Nash equilibrium, s^* , is positive. If we choose \bar{s} such that $(n+1)\beta\bar{s} > \alpha - n \min_i c_i + (n-1) \max_i c_i$, then $s_i^* \in (0, \bar{s})$ for all i .

game and a zero-sum game, as the following two payoffs show

$$\begin{pmatrix} (\alpha - \beta s_1)s_1 - c_1(s_1) & +(\alpha - \beta s_2)s_2 - c_2(s_2) & +(\alpha - \beta s_3)s_3 - c_3(s_3) \\ (\alpha - \beta s_1)s_1 - c_1(s_1) & +(\alpha - \beta s_2)s_2 - c_2(s_2) & +(\alpha - \beta s_3)s_3 - c_3(s_3) \\ (\alpha - \beta s_1)s_1 - c_1(s_1) & +(\alpha - \beta s_2)s_2 - c_2(s_2) & +(\alpha - \beta s_3)s_3 - c_3(s_3) \end{pmatrix} \quad (9)$$

and

$$\begin{pmatrix} (\alpha - \beta s_1)s_1 - c_1(s_1) - [(\alpha - \beta s_2)s_2 - c_2(s_2)] \\ (\alpha - \beta s_2)s_2 - c_2(s_2) - [(\alpha - \beta s_3)s_3 - c_3(s_3)] \\ (\alpha - \beta s_3)s_3 - c_3(s_3) - [(\alpha - \beta s_1)s_1 - c_1(s_1)] \end{pmatrix}. \quad (10)$$

The payoffs in (9) and (10) are payoffs for a common interest game and a zero-sum game, respectively. They are obtained from the self-interaction term by adding payoffs that do not depend on the player's own strategy and thus they are strategically equivalent (see Definition 2.2).

In a similar way, the payoff component describing the interactions between players 1 and 2 is strategically equivalent to the payoff for a common interest game and the payoff for a zero-sum game. For example,

$$\begin{pmatrix} \beta s_1 s_2 \\ \beta s_1 s_2 \\ 0 \end{pmatrix}, \begin{pmatrix} \beta s_1 s_2 \\ \beta s_1 s_2 \\ \beta s_1 s_2 \end{pmatrix}, \text{ and } \begin{pmatrix} \beta s_1 s_2 \\ \beta s_1 s_2 \\ -2\beta s_1 s_2 \end{pmatrix} \quad (11)$$

are all strategically equivalent. A similar computation holds for the last two terms in equation (8) involving the interactions between players 1 and 3 as well as between players 2 and 3. As a consequence, the quasi-Cournot oligopoly game is strategically equivalent to both a common interest game and a zero-sum game.

Example 2 (Zero-sum equivalent games: Contest games).

Our next example is a contest game (Konrad, 2009), which is a zero-sum equivalent game. Consider an n -player game where the payoff function for the i -th player is defined by

$$f^{(i)}(s) = p^{(i)}(s_i, s_{-i})v - c_i(s_i) \text{ for } i = 1, \dots, n, \quad (12)$$

where $\sum_i p^{(i)}(s) = 1$ and $p^{(i)}(s) \geq 0$ for all $s \geq 0$, $c_i(0) = 0$, $c_i(\cdot)$ is continuous, increasing and convex and $v > 0$. First, f is a zero-sum equivalent game, since w ,

defined by

$$w^{(i)} = (p^{(i)}(s_i, s_{-i}) - \frac{1}{n})v - \frac{1}{n-1} \sum_{j \neq i} (c_i(s_i) - c_j(s_j))$$

for all i , is strategically equivalent to f and is itself a zero-sum game. In Section 3, we will show that the following function, $\Phi(s)$, plays an important role in studying the equilibrium property of zero-sum equivalent games:

$$\Phi(s) := \max_{t \in S} \sum_{i=1}^n w^{(i)}(t_i, s_{-i}) \quad (13)$$

for $s \in S$. To illustrate our result, suppose that $f^{(i)}$'s are given by a relatively simple form:

$$f^{(i)}(s) = \frac{s_i}{\sum_j s_j} v - c_i s_i. \quad (14)$$

Then it is known that the game specified by (14) admits a unique Nash equilibrium (Szidarovszky and Okuguchi, 1997). From the first order condition of equation (14), we can find a best response $s_i^*(s_{-i})$

$$c_i(s_i^* + \sum_{l \neq i} s_l)^2 = v \sum_{l \neq i} s_l.$$

And via some computation, we find that

$$\Phi_f(s) = (\sum_i c_i)(\sum_l s_l) - 2 \sum_i \sqrt{c_i v} \sqrt{\sum_{l \neq i} s_l} + (n-1)v \quad (15)$$

which is strictly convex⁵. Later we show that if Φ_f for zero-sum equivalent game f is strict convex, then f has a unique Nash equilibrium (Lemma 3.2 and Proposition 3.1). This method of using the function Φ in equation (13), differently from the existing

⁵In fact, the best response function accounting for the boundary conditions, $s_i \geq 0$, is given by

$$s_i^* = \begin{cases} 0 & \text{if } \frac{v}{c_i} \leq \sum_{l \neq i} s_l \\ \sqrt{\frac{v}{c_i}} \sqrt{\sum_{l \neq i} s_l} - \sum_{l \neq i} s_l & \text{otherwise} \end{cases}.$$

It can be shown that Φ_f defined for this function achieves the same minimum as the one in equation (15).

analysis relying on the special properties of the payoff functions (Szidarovszky and Okuguchi, 1997; Cornes and Hartley, 2005), provide an alternative way of showing the existence and uniqueness of equilibrium, hence can be extended to more general settings.

Example 3 (Bayesian games). A Bayesian game with finite type spaces can be viewed as a normal form game by defining a player with a specific type as a new player (see the type-agent representation in Myerson (1997)). More specifically, consider a two-player game, for which player 2 has two types, τ_1 and τ_2 :

		player 2: τ_1				player 2: τ_2	
		s'_1	s'_2			s'_1	s'_2
$(g^{(1)}, g^{(2)}) :=$	s_1	a, a	$0, 0$	$(h^{(1)}, h^{(2)}) :=$	s_1	a, b	$0, 0$
	s_2	$0, 0$	b, b		s_2	$0, 0$	b, a

where $(g^{(1)}, g^{(2)})$ and $(h^{(1)}, h^{(2)})$ are the payoff functions, given types τ_1 and τ_2 , respectively. In this example, player 1 is not sure whether her partner has common interests or conflicting interests. Suppose that τ_1 and τ_2 occur with probabilities p and $1 - p$, respectively. Then, we can introduce a new game in which players 1', 2', and 3' are player 1 in the original game, player 2 with τ_1 , and player 2 with τ_2 , respectively. The payoff function, f , for the new game is defined as follows :

$$f(s) = \begin{pmatrix} f^{(1')}(s_1, s_2, s_3) \\ f^{(2')}(s_1, s_2, s_3) \\ f^{(3')}(s_1, s_2, s_3) \end{pmatrix} = \begin{pmatrix} g^{(1)}(s_1, s_2) \\ g^{(2)}(s_1, s_2) \\ 0 \end{pmatrix} p + \begin{pmatrix} h^{(1)}(s_1, s_3) \\ 0 \\ h^{(2)}(s_1, s_3) \end{pmatrix} (1 - p).$$

Here, the payoff function is based on the players' *ex-ante* expected payoffs. That is, the original player 1 *ex-ante* expects $g^{(1)}p + h^{(1)}(1 - p)$, and the original player 2 *ex-ante* expects the sum of the payoffs of players 2' and 3'. Then, using manipulations similar to those in the Cournot example, we see that f is strategically equivalent to a zero-sum game. The reason is that each player's payoff function is the sum of the component games played among a subset of players (e.g., $f^{(1')} = g^{(1)}p + h^{(1)}(1 - p)$). In addition, we can show that if $g = (g^{(1)}, g^{(2)})$ and $h = (h^{(1)}, h^{(2)})$ are potential games, which is straightforward to do, then f is itself a potential game (see Proposition 3.4). Therefore, the Bayesian game given in this example is a potential game and is strategically equivalent to a zero-sum game. Later, we show that every Bayesian game, embedded as a normal form game with more players, is strategically equivalent to a zero-sum game in the resulting space of extended normal form games.

2.3. Preliminary results

For the purpose of exposition we present some preliminary results, some of which are known and elementary, including some extensions of results from finite games to continuous strategy games. The advantage of a Hilbert space structure is the natural concept of orthogonality in terms of the inner product, $\langle f, g \rangle$, in (2): we say that f and g are orthogonal if $\langle f, g \rangle = 0$. In the context of game theory there are two important orthogonality relations: (i) common interest games and zero-sum games are orthogonal (Kalai and Kalai (2010)) and (ii) non-strategic games and normalized games are orthogonal (Hwang and Rey-Bellet (2011); Candogan et al. (2011)). We first state these two facts and explain their meanings and consequences.

Proposition 2.1. *We have the decompositions of the space of all games:*

$$\mathcal{L} = \mathcal{C} \oplus \mathcal{Z} \tag{16}$$

$$\mathcal{L} = \mathcal{N} \oplus \mathcal{E} \tag{17}$$

where the direct sum \oplus means that every game has a **unique** decomposition into a sum of two orthogonal components.

Proof. See Corollary in B.1 and see Appendix A for a formal definition of direct sums. \square

The first orthogonality relation in (16) implies that common interest games and zero-sum games are orthogonal to each other, that is

$$\text{for all } f \in \mathcal{C} \text{ and } g \in \mathcal{Z}, \langle f, g \rangle = 0, \tag{18}$$

which is denoted by $\mathcal{C} \perp \mathcal{Z}$ (this property is easy to check, using Fubini's theorem). Moreover, any given game $f \in \mathcal{L}$ can be uniquely written as $f = c + z$ with $c \in \mathcal{C}$ and $z \in \mathcal{Z}$. Hence, f is a common interest game if and only if its zero-sum component is $z = 0$, and f is a zero-sum game if and only if its common interest component is $c = 0$. In this way, the concept of orthogonal projections allows us to identify and separate the different components of a game, representing common and conflicting interests, respectively.

To understand the second orthogonality relation, note the following simple, but useful, characterization for a non-strategic game: a function does not depend on a

variable if and only if the value of the integral of the function with respect to that variable gives the same value:

Lemma 2.1. *A game f is a non-strategic game if and only if*

$$f^{(i)}(s) = \frac{1}{m_i(S_i)} \int f^{(i)}(t_i, s_{-i}) dm_i(t_i) \text{ for all } i, \text{ for all } s. \quad (19)$$

Proof. Suppose that f satisfies (19). Then, clearly, $f^{(i)}$ does not depend on s_i for all i . Now let $f \in \mathcal{E}$. Then there exist ζ such that $f^{(i)}(s) = \zeta^{(i)}(s_{-i})$ for all s , which does not depend on s_i , for all i . Thus, by integrating, we see that (19) holds. \square

Using the definition of normalized games in (3) and the characterization for non-strategic games in (19), we can verify that

$$\text{for all } f \in \mathcal{E} \text{ and } g \in \mathcal{N}, \langle f, g \rangle = 0 \quad (20)$$

which illustrates the decomposition in (17). Thus, any game payoff f can be uniquely decomposed into $n \in \mathcal{N}$ and $e \in \mathcal{E}$. The game n is called a normalized game since any game f can be normalized such that it is equivalent to a game with a no non-strategic component, i.e.,

$$f \sim f - \left(\frac{1}{m_1(S_1)} \int f^{(1)}(s_1, \cdot) dm_1(s_1), \dots, \frac{1}{m_n(S_n)} \int f^{(n)}(s_n, \cdot) dm_n(s_n) \right). \quad (21)$$

To identify a potential game $(\mathcal{C} + \mathcal{E}, \text{Definition 2.3})$, it is natural to examine how any given game is “close” to a potential game by decomposing it into a potential component and a component which fails to be a potential game, that is which is orthogonal to a potential game. This leads to the decomposition in (22).

Proposition 2.2. *We have the following decomposition:*

$$\mathcal{L} = (\mathcal{C} + \mathcal{E}) \oplus (\mathcal{N} \cap \mathcal{Z}). \quad (22)$$

Proof. For finite strategy games, this result easily follows from the decomposition results of the Hilbert space, presented in Proposition A.2. For continuous strategy games, the space $(\mathcal{C} + \mathcal{E})$ has to be closed, which we proved in Proposition B.1. \square

The decomposition in (22) shows that every game can be uniquely decomposed into a potential component and a normalized zero-sum component. Examples of

normalized zero-sum games include the Rock-Paper-Scissors games and Matching Pennies games, as explained in the introduction. The decomposition in (22) can be regarded as a continuum version of the finite game decomposition in Candogan et al. (2011). However, the difference between the two decompositions is that they coincide only when the number of strategies of games is the same⁶ (see Section 6 for more precise relationships).

2.4. Main decomposition results

Notice that by Definition 2.4 the space of all zero-sum equivalent potential games is given by

$$\mathcal{B} = (\mathcal{C} + \mathcal{E}) \cap (\mathcal{Z} + \mathcal{E}).$$

To understand the orthogonal complement of $\mathcal{Z} + \mathcal{E}$, we observe that $\mathcal{Z} \perp \mathcal{C}$ and $\mathcal{E} \perp \mathcal{N}$ from Proposition 2.1. This means that a game which both belongs to a common interest game (\mathcal{C}) and a normalized game (\mathcal{N}) is an element of the orthogonal complements of $\mathcal{Z} + \mathcal{E}$; i.e., normalized common interest games are the orthogonal complements of $\mathcal{Z} + \mathcal{E}$. More precisely,

$$\mathcal{N} \cap \mathcal{C} \subset (\mathcal{Z} + \mathcal{E})^\perp, \quad (23)$$

where $(A)^\perp := \{f \in \mathcal{L} : \langle f, g \rangle = 0 \text{ for all } g \in A\}$. The non-trivial, converse inclusion of (23) involves some technical issues for continuous strategy games. In particular, we need to show that $\mathcal{Z} + \mathcal{E}$ is a closed subspace of the Hilbert space, \mathcal{L} (Proposition B.1). We then obtain our first main result (Theorem 2.1 (i)).

To understand the orthogonal complement of \mathcal{B} , we again use the fact that zero-sum games are orthogonal to common interest games and the fact that normalized games are orthogonal to non-strategic games, obtaining that

$$(\mathcal{N} \cap \mathcal{C}) \subset \mathcal{B}^\perp \text{ and } (\mathcal{N} \cap \mathcal{Z}) \subset \mathcal{B}^\perp \quad (24)$$

which, in turn, implies

$$(\mathcal{N} \cap \mathcal{C}) \oplus (\mathcal{N} \cap \mathcal{Z}) \subset \mathcal{B}^\perp. \quad (25)$$

⁶In fact the harmonic games in Candogan et al. (2011) are our normalized zero-sum games only when all the players have the same number of strategies, as in the case of the Rock-Paper-Scissors and Matching Pennies games.

$\mathcal{N} \cap \mathcal{C}$	$\mathcal{Z} + \mathcal{E}$		Theorem 3.1 (i)
$\mathcal{N} \cap \mathcal{C}$	\mathcal{B}	$\mathcal{N} \cap \mathcal{Z}$	Theorem 3.1 (ii)
$\mathcal{C} + \mathcal{E}$		$\mathcal{N} \cap \mathcal{Z}$	Proposition 3.2

Figure 1: **Relationships among decompositions.** Here, recall that $\mathcal{B} = (\mathcal{C} + \mathcal{E}) \cap (\mathcal{Z} + \mathcal{E})$.

Again, the converse inclusion is based on the facts that the spaces of potential games and zero-sum equivalent games are closed. The proofs of all these facts yield result (ii) in Theorem 2.1. We now stand ready to state our main results (see Figure 1).

Theorem 2.1 (Decompositions involving zero-sum equivalent games and zero-sum equivalent potential games). *We have the following two decompositions:*

$$\begin{aligned}
(i) \quad \mathcal{L} &= (\mathcal{N} \cap \mathcal{C}) \oplus (\mathcal{Z} + \mathcal{E}) \\
(ii) \quad \mathcal{L} &= (\mathcal{N} \cap \mathcal{C}) \oplus (\mathcal{N} \cap \mathcal{Z}) \oplus \mathcal{B}
\end{aligned}$$

Proof. We show that $\mathcal{Z} + \mathcal{E}$ and $\mathcal{C} + \mathcal{E}$ are closed in Proposition B.1. Then from this, $\mathcal{B} = (\mathcal{C} + \mathcal{E}) \cap (\mathcal{Z} + \mathcal{E})$ is closed as well. The decompositions in (i) and (ii) then follow from general decomposition results for a Hilbert space presented in Proposition A.2. \square

3. Zero-sum equivalent games ($\mathcal{Z} + \mathcal{E}$)

In this section, we start with the characterization of Nash equilibrium for any game in terms of an optimization problem (see Rosen (1965), Bregman and Fokin (1998), Myerson (1997), Barron (2008), Cai et al. (2015)). We then provide equilibrium characterizations of zero-sum equivalent games and conditions for zero-sum equivalent games.

3.1. Optimization problems for Nash equilibria

When we study Nash equilibria of finite games, we will consider both pure and mixed strategies. Thus, for finite games we let $\Delta_i = \{\sigma_i \in \mathbb{R}^{|S_i|} : \sum_{s_i \in S_i} \sigma_i(s_i) = 1, \sigma_i(s_i) \geq 0 \text{ for all } s_i\}$ with $\sigma_i(s_i)$ being the probability that player i uses strategy

s_i . We also extend the domain of the payoff f from S to $\Delta = \times_{i=1}^n \Delta_i$ by defining

$$f^{(i)}(\sigma) := \sum_{s \in S} f^{(i)}(s) \prod_k \sigma_k(s_k). \quad (26)$$

For continuous strategy games, we mainly consider the set of Nash equilibria in pure strategies. We consider a class of games for which each player's best response is well-defined and the payoff, when playing a best response, is finite: i.e., $\max_{s_i \in S_i} \{f^{(i)}(s_i, s_{-i})\}$ admits a solution and exists for all i . The following regularity condition ensures this.

Condition (R). *Suppose that S_i is non-empty, convex and compact and $f^{(i)}$ is upper-semi continuous for all i .*

The Nash equilibria for two strategically equivalent games, by definition, are the same. That is, if f and g satisfy condition (R) and are strategically equivalent, then the set of all Nash equilibria for f is equal to the set of all Nash equilibria for g .

To obtain an optimization characterization for the Nash equilibria of game f , we first note that s^* is a Nash equilibrium if and only if

$$\max_{t \in S} \sum_{i=1}^n f^{(i)}(t_i, s_{-i}^*) = f^{(i)}(s^*) \iff \max_{t \in S} \sum_{i=1}^n (f^{(i)}(t_i, s_{-i}^*) - f^{(i)}(s^*)) = 0 \quad (27)$$

To obtain further characterization, we let

$$\Phi_f(s) := \max_{t \in S} \sum_{i=1}^n (f^{(i)}(t_i, s_{-i}) - f^{(i)}(s)) \quad \text{for } s \in S \quad (28)$$

Then s^* is a Nash equilibrium if and only if $\Phi_f(s^*) = 0$ and also since

$$\max_{t \in S} \sum_{i=1}^n (f^{(i)}(t_i, s_{-i}) - f^{(i)}(s)) \geq 0$$

for all s , $\Phi_f(s) \geq 0$ for all $s \in S$. Thus s^* is a Nash equilibrium if and only if

$$0 = \Phi(s^*) = \min_s \Phi(s) \quad (29)$$

	Properties	Example
Zero-sum equivalent games	Convexity/ uniqueness of NE under some conditions	contest games (C) quasi-Cournot games (C)
Zero-sum equivalent potential games	Two-player games: dominant strategy NE Multilateral symmetric games	Prisoner's Dilemma (F) quasi-Cournot games
Normalized zero-sum games	Unique uniform mixed strategy NE	Rock-Paper-Scissors games (F) Matching Pennies games (F)
Normalized common interest games	Uniform mixed strategy NE	Coordination games

Table 3: **Summary of equilibrium characterizations for game.** In the table, (C) and (F) mean continuous strategy games and finite games, respectively.

In particular, if game f admit a Nash equilibrium, then the minimizer of $\Phi(s)$ becomes a Nash equilibrium. We summarize these observations in the following lemma.

Lemma 3.1. *Assume Condition **(R)** and let Φ_f be given by (28). Then, s^* is a minimizer to (29) and $\Phi_f(s^*) = 0$ if and only if s^* is a Nash equilibrium.*

Proof. This easily follows from the discussion before this lemma. \square

3.2. Equilibrium characterizations for zero-sum equivalent games

Using the results from the previous section, we provide equilibrium characterization for zero-sum equivalent games (see Table 3 for a summary). Recall that

$$\mathcal{N} \cap \mathcal{Z} \subset \mathcal{Z} \subset \mathcal{Z} + \mathcal{E}.$$

Thus, the properties that hold for $\mathcal{Z} + \mathcal{E}$ are most general since these properties hold for all normalized zero-sum games ($\mathcal{N} \cap \mathcal{Z}$), zero-sum games (\mathcal{Z}) and zero-sum equivalent games ($\mathcal{Z} + \mathcal{E}$). The results in this section thus automatically carry over into smaller classes.

If f is a zero-sum equivalent game such that $f = w + h$, then

$$\Phi_f(s) = \max_{t \in S} \sum_{i=1}^n w^{(i)}(t_i, s_{-i}) \quad \text{for } s \in S$$

which we presented in Section 2.2. Furthermore, by imposing some conditions for the payoff function, w , and using Φ_f function, we can derive useful characterizations for Nash equilibrium of zero-sum equivalent games. To do this, we recall the following facts, whose proofs are elementary.

Lemma 3.2. *Suppose that S is convex and that $\Phi_f(s)$ achieves a minimum.*

- (i) *If $\Phi_f(s)$ is convex, the set of minimizers is convex.*
- (ii) *If $\Phi_f(s)$ is strictly convex, the minimizer is unique.*

Proof. The proofs are elementary and hence are omitted. □

To derive characterizations for Nash equilibrium, we will impose that $w^{(i)}(s_i, s_{-i})$ is (strictly) convex in s_{-i} . If $w^{(i)}(s_i, s_{-i})$ is convex in s_{-i} , $w^{(i)}(s_i, s_{-i})$ is convex in s_j for $j \neq i$ and thus $-w^{(i)}(s_i, s_{-i})$ is concave in s_j for $j \neq i$. Also, since $w = (w^{(1)}, w^{(2)}, \dots, w^{(n)})$ is a zero-sum game, we have

$$w^{(i)}(s_i, s_{-i}) = - \sum_{l \neq i} w^{(l)}(s_l, s_{-l}).$$

Since $-w^{(l)}(s_l, s_{-l})$ is concave in s_i and the sum of all concave functions is again concave, $w^{(i)}$ is concave in s_i for all i , called a concave game.⁷ Rosen (1965) shows that if a game is a concave game, there exists a Nash equilibrium.

As we will show shortly in Corollary 3.1, two-player finite zero-sum equivalent games typically have a unique equilibrium. Proposition 3.1 below shows that, in general, the set of Nash equilibria for n -player zero-sum equivalent games is convex under some plausible conditions. Since a convex set in \mathbb{R}^d is connected, the convex set of Nash equilibria for zero-sum equivalent games is a connected set, generalizing the property of uniqueness.

Proposition 3.1 (Nash equilibrium for zero-sum equivalent games). *Suppose that f is a zero-sum equivalent game, where $f = w + h$; w is a zero-sum game and h is a non-strategic game. Suppose that Condition **(R)** is satisfied.*

- (i) *If $w^{(i)}(s_i, s_{-i})$ is convex in s_{-i} for all s_i for all i , there exists a Nash equilibrium and the set of Nash equilibria is convex.*
- (ii) *If $w^{(i)}(s_i, s_{-i})$ is strictly convex in s_{-i} for all s_i for all i , there exists a unique Nash equilibrium for f .*

Proof. We show (ii) follows similarly. Let i and s_i be fixed. Then from the discussion before the proposition, $w^{(i)}$ is concave in s_i for all i . Thus there exists a Nash equilibrium. We next show that $\Phi(s) = \sum_i \max_{s_i \in S_i} w^{(i)}(s_i, s_{-i})$ is strictly

⁷We thanks for an anonymous referee for pointing this.

convex. Let $t', t'' \in S$ be given. Then $u', u'' \in S$ be given such that $w^{(i)}(u'_i, t'_{-i}) = \max_{s_i \in S_i} w^{(i)}(s_i, t'_{-i})$ and $w^{(i)}(u''_i, t''_{-i}) = \max_{s_i \in S_i} w^{(i)}(s_i, t''_{-i})$ for all i . Let $\alpha \in (0, 1)$ and t^* be such that $w^{(i)}(t^*_i, ((1-\alpha)t' + \alpha t'')_{-i}) = \max_{s_i \in S_i} w^{(i)}(s_i, ((1-\alpha)t' + \alpha t'')_{-i})$ for all i . Then we have

$$\begin{aligned} (1-\alpha)\Phi(t') + \alpha\Phi(t'') &= (1-\alpha) \sum_i w^{(i)}(u'_i, t'_{-i}) + \alpha \sum_i w^{(i)}(u''_i, t''_{-i}) \\ &\geq (1-\alpha) \sum_i w^{(i)}(t^*_i, t'_{-i}) + \alpha \sum_i w^{(i)}(t^*_i, t''_{-i}) > \sum_i w^{(i)}(t^*_i, (1-\alpha)t'_{-i} + \alpha t''_{-i}) \\ &= \Phi((1-\alpha)t' + \alpha t''). \end{aligned}$$

Thus from Lemma 3.2, the minimizer of Φ_f is unique. Since the Nash equilibrium is a minimizer of Φ_f , the Nash equilibrium is unique. \square

The idea behind Proposition 3.1 is that if $w^{(i)}$ is convex in s_{-i} , then $\Phi_f(s)$ is convex. The convexity of $\Phi_f(s)$ is analogous to the convexity of the profit function in a basic microeconomics context (see Figure 2). To explain this, consider a two-player game and assume that $w^{(1)}(s_1, s_2)$ is linear (hence convex) with respect to s_2 ; thus, $w^{(1)}(s_1, s_2) = g(s_1) + \alpha s_2$ for some $\alpha \in \mathbb{R}$. Let s_1^0 be the best response against s_2^0 that yields the maximum payoff to player 1. If we define $\tilde{w}^{(1)}(s_2) := g(s_1^0) - \alpha s_2$, then

$$\tilde{w}^{(1)}(s_2) = \begin{cases} w^{(1)}(s_1^0, s_2^0) = \max_{s_1} w^{(1)}(s_1, s_2^0), & \text{if } s_2 = s_2^0 \\ w^{(1)}(s_1^0, s_2) \leq \max_{s_1} w^{(1)}(s_1, s_2), & \text{if } s_2 \neq s_2^0. \end{cases}$$

That is, the payoff from adopting the best response ($\max_{s_1} w^{(1)}(s_1, s_2^0)$) must be at least as large as the payoff from adopting the non-best response ($\tilde{w}^{(1)}(s_2)$). Since $\tilde{w}^{(1)}$ is linear and $\max_{s_1} w^{(1)}(s_1, s_2)$ lies above $\tilde{w}^{(1)}$ with passing through the only point $(s_2^0, w^{(1)}(s_1^0, s_2^0))$ (see Panel A, Figure 2), hence $\max_{s_1} w^{(1)}(s_1, s_2)$ is convex. Clearly, the same argument holds when $w^{(1)}(s_1, s_2)$ is strictly convex with respect to s_2 (see Panel B, Figure 2). Then the convexity of $\Phi_f(s)$ follows from the sum of convex functions remaining convex. Rosen (1965) also provides some condition for the uniqueness of the Nash equilibrium of a concave game and we compare our condition and Rosen's condition in Appendix D.

To further explore the consequences of Proposition 3.1 for finite games, we focus on a class of games which is non-degenerate. There are several notions of non-degeneracy in finite games, depending on contexts and problems—such as equilib-

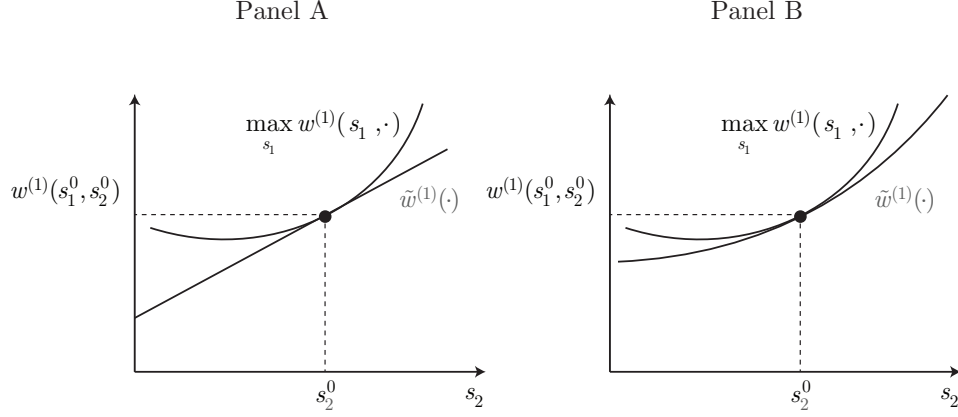


Figure 2: **Illustration of Proposition 3.1.** Panel A shows the case when $\tilde{w}^{(1)}(\cdot)$ is linear (hence convex), while Panel B shows the case when $\tilde{w}^{(1)}(\cdot)$ is strictly convex.

rium characterizations and classifications of dynamics.⁸ Since we wish to study the equilibrium properties of games, we are interested in a class of games that [Wilson \(1971\)](#) identified—namely games with an odd (hence finite) number of equilibrium.

Condition (N). *We call a finite game non-degenerate if it has a finite number of Nash equilibria.*

Next we restrict our attention further to the space of two-player games, often called bi-matrix games. Even though it is one of the simplest classes that we consider in the paper, in general it is generally acknowledged that even bi-matrix games are hard to solve ([Savani and von Stengel, 2006](#)). A straightforward consequence of Proposition 3.1 is that, generically, two-player zero-sum equivalent games have a *unique* Nash equilibrium.

Corollary 3.1 (two-player finite zero-sum equivalent games). *Suppose that f is a two-player finite zero-sum equivalent game. Then the set of Nash equilibria for f is convex. If f satisfies Condition (N), the Nash equilibrium is unique.*

⁸[Wu and Jiang \(1962\)](#) define an essential game—a game whose Nash equilibria all change only slightly against a smaller perturbation to the game and show that almost all finite games are essential; i.e., the set of all essential games is an open and dense subset of the space of games. [Wilson \(1971\)](#) introduces a non-degeneracy assumption regarding payoff matrices (more precisely tensors) and shows that almost all games have an odd (hence finite) number of Nash equilibria. In the context of evolutionary game theory, [Zeeman \(1980\)](#) also defines a stable game whose dynamic remains structurally unchanged against a small perturbation.

Proof. Let $f = w + h$, where h is a non-strategic game. Then, $w^{(1)}(\sigma_1, \sigma_2)$ is convex in σ_2 and $w^{(2)}(\sigma_1, \sigma_2)$ is convex in σ_1 . By Proposition 3.1, the set of Nash equilibria is convex. Suppose that f has two distinct Nash equilibria, ρ^* and σ^* , where $\rho^* \neq \sigma^*$. Then, for all $t \in (0, 1)$, $(1 - t)\rho^* + t\sigma^*$ is a Nash equilibrium since the set of Nash equilibria is convex. This contradicts Condition (N). \square

3.3. Conditions for zero-sum equivalent games

It is *a priori* not clear how to determine if a game is a potential game or not. To check this, some tests have been proposed to determine if a given game is a potential game (Monderer and Shapley, 1996; Ui, 2000; Sandholm, 2010a; Hino, 2011; Hwang and Rey-Bellet, 2015). Similarly zero-sum equivalent games are not always easily recognizable. We thus provide some conditions for zero-sum equivalent games. Recall that Monderer and Shapley (1996) provide an elegant characterization for potential games, often called the cycle condition, although the verification of this condition in practice may not be easily implementable (see Hino (2011)). Our first condition for zero-sum equivalent games is a direct analog of the cycle condition for potential games (Proposition 3.2).

To explain the idea behind this condition, recall that the Monderer and Shapley test for two-player games is given by

$$\begin{aligned} & f^{(1)}(s_1, s_2) - f^{(1)}(t_1, s_2) - f^{(1)}(s_1, t_2) + f^{(1)}(t_1, t_2) \\ &= f^{(2)}(s_1, s_2) - f^{(2)}(t_1, s_2) - f^{(2)}(s_1, t_2) + f^{(2)}(t_1, t_2) \end{aligned} \quad (30)$$

for all $s_1, t_1 \in S_1$, $s_2, t_2 \in S_2$. We will show that the condition for zero-sum equivalent games is similarly given by

$$\begin{aligned} & f^{(1)}(s_1, s_2) - f^{(1)}(t_1, s_2) - f^{(1)}(s_1, t_2) + f^{(1)}(t_1, t_2) \\ &+ f^{(2)}(s_1, s_2) - f^{(2)}(t_1, s_2) - f^{(2)}(s_1, t_2) + f^{(2)}(t_1, t_2) = 0 \end{aligned} \quad (31)$$

for all $s_1, t_1 \in S_1$, $s_2, t_2 \in S_2$. To see that the condition in (31) is a necessary condition for zero-sum equivalent games, let $f = w + h$, where w is a zero-sum game and h is a non-strategic game. Obviously, w satisfies (31) (e.g., $w^{(1)}(s_1, s_2) + w^{(2)}(s_1, s_2) = 0$). Also h satisfies (31) too; e.g., $h^{(1)}(s_1, s_2) - h^{(1)}(t_1, s_2) = 0$ since $h^{(1)}$ does not depend on s_1, t_1 (a non-strategic game). To see the sufficiency of the condition in (31), first we fix $t = (t_1, t_2)$ and use $x = (x_1, x_2)$ as variables. Let g be a normalized

common interest game in $\mathcal{N} \cap \mathcal{C}$. Then there exists v such that $g = (v, v)$ and $\int v(x) dm_i(x_i) = 0$ for $i = 1, 2$. If $f^{(1)}(x_1, x_2) = f^{(1)}(t_1, x_2) + f^{(1)}(x_1, t_2) - f^{(1)}(t_1, t_2)$ holds, then

$$\begin{aligned}
& \int g^{(1)}(x_1, x_2)(f^{(1)}(x_1, x_2)) dm(x) \\
&= \int v(x_1, x_2)(f^{(1)}(t_1, x_2) + f^{(1)}(x_1, t_2) - f^{(1)}(t_1, t_2)) dm(x) \\
&= \int v(x_1, x_2)f^{(1)}(t_1, x_2) dm(x) + \int v(x_1, x_2)f^{(1)}(x_1, t_2) dm(x) - \int v(x_1, x_2)f^{(1)}(t_1, t_2) dm(x) \\
&= 0
\end{aligned} \tag{32}$$

where the last line follows because of normalization, $\int v(x) dm_i(x) = 0$ for $i = 1, 2$. Thus, if f satisfies (31), then

$$\begin{aligned}
& f^{(1)}(x_1, x_2) + f^{(2)}(x_1, x_2) \\
&= f^{(1)}(t_1, x_2) + f^{(1)}(x_1, t_2) - f^{(1)}(t_1, t_2) + f^{(2)}(t_1, x_2) + f^{(2)}(x_1, t_2) - f^{(2)}(t_1, t_2)
\end{aligned}$$

holds and we can compute similarly to (32) and find that

$$\langle g, f \rangle = \int v(x)(f^{(1)}(x) + f^{(2)}(x)) dm(x) = 0$$

This shows that $g \perp f$ and our decomposition in Theorem 2.1 (i) concludes that f is a zero-sum equivalent.

To state a general n -player version, we need the following notations: Let $a_1, b_1 \in S_1, \dots, a_n, b_n \in S_n$ and let

$$\begin{aligned}
\mathcal{S}(a, b) &:= \{s = (s_1, \dots, s_n) : s_i = a_i \text{ or } b_i \text{ for all } i\} \text{ and} \\
\#(s) &:= \text{the number of } a_i\text{'s in } s
\end{aligned} \tag{33}$$

Proposition 3.2 (zero-sum equivalent games). *A game f is a zero-sum equivalent game if and only if*

$$\sum_{i=1}^n \sum_{s \in \mathcal{S}(a, b)} (-1)^{\#(s)} f^{(i)}(s) = 0 \tag{34}$$

for all $a_1, b_1 \in S_1, \dots, a_n, b_n \in S_n$.

Proof. See Appendix C. □

For the class of bi-matrix games (i.e., two-player finite strategy games), Hofbauer and Sigmund (1998) give the finite game version of (31) (Exercise 11.2.9 on page 131). When two-player games are symmetric (hence satisfying the condition that $f^{(1)}(s_1, s_2) = f^{(2)}(s_2, s_1)$; for an example, see games in Table 1)⁹, the condition in (31) (or Proposition 3.2) becomes even simpler:

Corollary 3.2. *Consider two-player symmetric games: i.e., $f^{(1)}(s_1, s_2) = f^{(2)}(s_2, s_1)$ for all s_1, s_2 . Then $f = (f^{(1)}, f^{(2)})$ is a zero-sum equivalent game if and only if*

$$f^{(1)}(s, t) - f^{(1)}(t, t) + f^{(1)}(t, s) - f^{(1)}(s, s) = 0 \text{ for all } s, t \in S_1. \quad (35)$$

Proof. See Appendix C. □

Although Proposition 3.2 provides a condition for zero-sum equivalent games analogous to a cycle condition for potential games in Monderer and Shapley (1996), checking this condition may be more difficult in practice¹⁰. There are also alternative tests for zero-sum equivalent games, called an integral test and a derivative test, with which we can determine whether a given game is zero-sum equivalent or not (Hwang and Rey-Bellet, 2015).

⁹Here, the first arguments of $f^{(1)}$ and $f^{(2)}$ are the strategies of player 1 and the second arguments of them are the strategies of player 2. Because of abuse of notation, $f^{(i)}(s_1, s_2, \dots, s_n) = f^{(i)}(s_i, s_{-i})$, the meaning of the first argument of $f^{(2)}$ may create confusion.

¹⁰To compare the computational burdens of Monderer and Shapley's test and the zero-sum equivalent test, consider an n -player finite game for which each player has the same number of strategies, say s . Then, we can find the following requirements for the two tests:

	numbers of Eqs to be checked
Monderer and Shapley test	$\binom{n}{2} \binom{s}{2} \binom{s}{2} \times s^{n-2}$
Zero-sum equivalent game test	$\binom{s}{2}^n$

Thus, when n is small, the zero-sum equivalent game test involves less number of equations to be checked than for Monderer and Shapley's test, but when n is large, the opposite holds. In general, when n is large, Monderer and Shapley's test algorithm is of the order $O(s^{(n+2)})$, while the zero-sum equivalent is of the order $O(s^{2n})$.

Proposition 3.3. *Consider an n -player game, $f = (f^{(1)}, \dots, f^{(n)})$. Suppose that for all i ,*

$$f^{(i)}(s) = \frac{1}{m_l(S_l)} \int f^{(i)}(t_l, s_{-l}) dm_l(t_l),$$

for some l . Then, f is strategically equivalent to a zero-sum game.

Proof. See Appendix C. □

Using Proposition 3.3, we study Bayesian games with finite strategy spaces. For simplicity, we consider two-player games with the same number of possible types for both players. That is, player α 's type, τ_1 , takes the values $\tau_1^{(i)}$ with probabilities p_i , for $i = 1, \dots, k$. Similarly, player β 's type, τ_2 , takes the values $\tau_2^{(i)}$ with probabilities q_i , for $i = 1, \dots, k$. Suppose that a payoff function for a given Bayesian game is

$$f(s, \tau) = (f^{(1)}(s_\alpha, s_\beta, \tau_1, \tau_2), f^{(2)}(s_\alpha, s_\beta, \tau_1, \tau_2)). \quad (36)$$

We consider a $2k$ -player game in which player i is player α with type $\tau_1^{(i)}$ if $i \leq k$, and is player β with type $\tau_2^{(i-k)}$ if $i \geq k+1$. We define a new payoff function, π , for the extended $2k$ -player game:

$$\pi^{(i)}((s_1, \dots, s_k), (s_{k+1}, \dots, s_{2k})) = \begin{cases} \sum_{l=k+1}^{2k} f^{(1)}(s_i, s_l, \tau_1^{(i)}, \tau_2^{(l)}) p_i q_l & \text{if } i \leq k \\ \sum_{l=1}^k f^{(2)}(s_l, s_i, \tau_1^{(l)}, \tau_2^{(i)}) p_l q_i & \text{if } i \geq k+1 \end{cases}. \quad (37)$$

We then have the following characterizations for Bayesian games, which shows that the set of all Bayesian games is contained in $\mathcal{Z} + \mathcal{E}$.

Proposition 3.4 (Bayesian games). *Consider a two-player Bayesian game with finite type spaces as defined in (36). We have the following characterizations:*

- (i) *The extended normal form game in (37) is strategically equivalent to a zero-sum game.*
- (ii) *If the underlying game is a potential game for all possible types, then the extended normal form game in (37) is a potential game.*

4. Zero-sum equivalent potential games: $\mathcal{B} = (\mathcal{Z} + \mathcal{E}) \cap (\mathcal{C} + \mathcal{E})$

4.1. Representation of n -player games

We denote by $\zeta_l : S \rightarrow \mathbb{R}$ a function that does not depend on s_l ; thus, $\zeta_l(s) = \zeta_l(s_{-l})$ for all s . We define the following special class of games, an example of which

is the quasi-Cournot model in Section 2.2.

Definition 4.1. In the space of n player games, the subspace of $n - 1$ *multilateral common interest games* is defined by

$$\mathcal{D} = \{f \in \mathcal{L} : f^{(i)}(s) := \sum_{l \neq i} \zeta_l(s_{-l}) \text{ for all } i\}.$$

The class of $n - 1$ multilateral common interest games includes the set of bilateral symmetric games in Ui (2000). Conversely, $n - 1$ multilateral common interest games belong to a special class of games with interaction potentials in Ui (2000). We explain the relationship between Ui's results and ours in Section 6 in more detail. For example, for a 3-multilateral common interest game (as a 4-player game), the payoff function for player 1 is given by

$$f^{(1)}(s_1, s_2, s_3, s_4) = \zeta_2(s_1, s_3, s_4) + \zeta_3(s_1, s_2, s_4) + \zeta_4(s_1, s_2, s_3)$$

and each term $f^{(i)}$ depends only on, at most, $n - 1$ variables.

In Section 2.2, we demonstrated that the quasi-Cournot model (a 2-multilateral common interest game) is a potential game which is also strategically equivalent to a zero-sum game in the case of three players (see equations (9), (10), and (11)). Proposition 4.1 below shows that, in general, the class of zero-sum equivalent potential games, \mathcal{B} , coincides with the class of $n - 1$ multilateral common interest games, \mathcal{D} . For two-player games, the games in \mathcal{D} have payoffs of the form $\zeta_2(s_1) + \zeta_1(s_2)$ and this kind of game generically has a dominant strategy, hence the choice of the name, \mathcal{D} (see Corollary 4.1).

Proposition 4.1. *We have the following characterizations.*

- (i) *Every zero-sum equivalent potential game is strategically equivalent to an $n - 1$ multilateral common interest game.*
- (ii) *Every $n - 1$ multilateral common interest game is strategically equivalent to both a zero-sum game and a common interest game, hence is a zero-sum equivalent potential game.*

That is,

$$\mathcal{B} = \mathcal{D} + \mathcal{E}$$

Proof. See Appendix C. □

If a game is a zero-sum equivalent potential game, it can be expressed, up to strategic equivalence, as a common interest game or a zero-sum game. One advantage of multilateral common interest games is that the multilateral components explicitly give expressions for equivalent common interest games (hence potential functions) and equivalent zero-sum games (hence Φ_f , in (28)), as the following proposition shows:

Proposition 4.2 (*n*-player zero-sum potential equivalent games). *An n -player game with payoff f is a zero-sum equivalent potential game if and only if*

$$(f^{(1)}, f^{(2)}, \dots, f^{(n)}) \sim \sum_{l=1}^n (\zeta_l, \zeta_l, \dots, \zeta_l) \quad (38)$$

$$\sim \sum_{i < j} (0, \dots, 0, \underbrace{-\zeta_i + \zeta_j}_{i\text{-th}}, 0, \dots, 0, \underbrace{\zeta_i - \zeta_j}_{j\text{-th}}, 0, \dots, 0), \quad (39)$$

where $\zeta_l(\cdot)$ does not depend on s_l .

Proof. See Appendix C. □

A similar expression to (38) for potential functions is in Ui (2000) (see the potential function in Theorem 3 in the cited paper). The first part (38) of Proposition 4.2 shows that n -player zero-sum equivalent potential games are those games in which every player simultaneously plays n games, each of which is an $(n - 1)$ -player common interest game, $\zeta_l(s_{-l})$ (including one in which the payoff does not depend on the player's own strategy). Alternatively, these kinds of games can be viewed as simultaneously playing dyadic zero-sum games, as the second part (39) of Proposition 4.2 shows. For example, when $n = 3$,

$$f \sim (\zeta_2 - \zeta_1, \zeta_1 - \zeta_2, 0) + (\zeta_3 - \zeta_1, 0, \zeta_1 - \zeta_3) + (0, \zeta_3 - \zeta_2, \zeta_2 - \zeta_3).$$

In the case of $n(\geq 3)$ -player games, because of the externality of strategic interactions, the payoff functions for the dyadic zero-sum games are generally affected by other players, as well as by those who are directly involved. For example,

$$(\zeta_2 - \zeta_1, \zeta_1 - \zeta_2, 0) = (\zeta_2(s_1, s_3) - \zeta_1(s_2, s_3), \zeta_1(s_2, s_3) - \zeta_2(s_1, s_3), 0).$$

Here, the payoff functions for players 1 and 2 are affected by the strategy choices of

player 3, as well as those of players 1 and 2.

4.2. Two-player games

We now turn our attention to two-player games in the class of zero-sum equivalent potential games. In this case, the $n - 1$ multilateral common interest games are of the form $(\zeta_2(s_1), \zeta_1(s_2))$. That is, a player's payoff depends only on her own strategy choices. Thus, an immediate consequence of this observation and Proposition 4.2 is as follows:

Corollary 4.1 (Two-player zero-sum equivalent potential games). *A two-player game with payoff $f = (f^{(1)}, f^{(2)})$ is a zero-sum equivalent potential game if and only if*

$$\begin{aligned} (f^{(1)}, f^{(2)}) &\sim (\zeta_1(s_2) + \zeta_2(s_1), \zeta_1(s_2) + \zeta_2(s_1)) \\ &\sim (-\zeta_1(s_2) + \zeta_2(s_1), \zeta_1(s_2) - \zeta_2(s_1)). \end{aligned} \quad (40)$$

Suppose that Condition **(R)** is satisfied. Then, $(s_1^*, s_2^*) \in (\arg \max_{s_1} \zeta_2(s_1), \arg \max_{s_2} \zeta_1(s_2))$ is a Nash equilibrium.

Proof. This immediately follows from Proposition 4.2. \square

Intuitively, when two players have both common and conflicting interests, the strategic interdependence effects completely offset each other as in the Prisoner's Dilemma game. If we consider a symmetric game in which $f^{(1)}(s_1, s_2) = f^{(2)}(s_2, s_1)$ for all s_1, s_2 (see footnote 9), condition (40) becomes even stronger:

Corollary 4.2. *Suppose that $f = (f^{(1)}, f^{(2)})$ is symmetric; i.e., $f^{(1)}(s_1, s_2) = f^{(2)}(s_2, s_1)$ for all s_1, s_2 . Then, f is a zero-sum equivalent potential game if and only if*

$$f^{(1)}(s_1, s_2) = \zeta_1(s_2) + \zeta_2(s_1) \text{ for some } \zeta_1 \text{ and } \zeta_2. \quad (41)$$

Proof. If f is a two-player game, then $f \in \mathcal{D} + \mathcal{E}$ if and only if

$$f^{(1)}(s_1, s_2) = \zeta_1(s_2) + \zeta_2(s_1) \text{ and } f^{(2)}(s_1, s_2) = \zeta_1'(s_2) + \zeta_2'(s_1) \quad (42)$$

for some ζ_1, ζ_2 and ζ_1', ζ_2' . Now, if f is symmetric, then clearly (41) holds if and only if (42) holds. \square

Using a different approach, [Duersch et al. \(2012\)](#) provide partial results of Corollary 4.2. They show that when a game is a two-player zero-sum game, condition (41) holds if and only if the game is a potential game. Our result also shows that every game described by (41) is a zero-sum equivalent potential game. In the context of population games, [Sandholm \(2010b\)](#) refers to a game strategically equivalent to (41) as a constant game and shows that a game is a constant game if and only if it is a potential game with a linear potential function (Proposition 3.2.16 in [Sandholm \(2010b\)](#)).

In the case of finite games, a stronger characterization than Corollary 4.1 is possible as follows.

Corollary 4.3. *Suppose that a two-player finite zero-sum equivalent potential game satisfies Condition (N). Then the game has a strictly dominant strategy Nash equilibrium.*

Proof. From the second part of Corollary 4.1, $(s_1^*, s_2^*) \in (\arg \max_{s_1} \zeta_2(s_1), \arg \max_{s_2} \zeta_1(s_2))$ is a Nash equilibrium. If there are two distinct maximizers, then since the set of maximizers is convex, there exist infinitely many Nash equilibria, contradicting Condition (N). Thus, the maximizer is unique and constitutes the strictly dominant Nash equilibrium. \square

5. Decomposition of normal form games into components with distinctive Nash equilibrium characterizations

In this section, we present the decomposition of a given game into components with different Nash equilibrium characterizations, such as the existence of a completely mixed strategy equilibrium and a strictly dominant strategy equilibrium. Before presenting the main results, we show that every normalized zero-sum game and common interest game possess a uniform mixed strategy Nash equilibrium.

Proposition 5.1 (Normalized zero-sum games and normalized common interest games). *Suppose that a game is a normalized zero-sum game or a common interest game. Then the uniform mixed strategy profile is always a Nash equilibrium.*

Proof. Recall from equation (5) that $d\sigma_i(s_i) = \frac{1}{m(S_i)} dm_i(s_i)$ is a uniform mixed strategy. We define a uniform mixed strategy profile as a product measure of uniform

mixed strategies: i.e.,

$$d\sigma(s) = \prod_i d\sigma_i(s_i).$$

Let i and s_i be fixed. We show that

$$f^{(i)}(s_i, \sigma_{-i}) = 0.$$

Then, the desired result follows since $f^{(i)}(s_i, \sigma_{-i}) = 0 = f^{(i)}(\sigma_i, \sigma_{-i})$ for all i and s_i ; hence, $f^{(i)}(\sigma_i, \sigma_{-i}) = \max_{s_i} f^{(i)}(s_i, \sigma_{-i})$ for all i . First, by the definition of the mixed strategy extension,

$$f^{(i)}(s_i, \sigma_{-i}) = \int_{s_{-i} \in S_{-i}} f^{(i)}(s_i, s_{-i}) \prod_{l \neq i} d\sigma_l(s_l).$$

If f is a normalized zero-sum game, then

$$f^{(i)}(s_i, \sigma_{-i}) = - \int_{s_{-i} \in S_{-i}} \sum_{j \neq i} f^{(j)}(s_j, s_{-j}) \prod_{l \neq i} d\sigma_l(s_l) = - \sum_{j \neq i} \int_{s_{-i} \in S_{-i}} f^{(j)}(s_j, s_{-j}) \prod_{l \neq i} d\sigma_l(s_l) = 0$$

where the last equality follows from the normalization, $\int_{s_l \in S_l} f^{(l)}(s_l, s_{-l}) d\sigma_l(s_l) = 0$ for all l and Fubini's Theorem. If f is a common interest game, then similarly

$$f^{(i)}(s_i, \sigma_{-i}) = \int_{s_{-i} \in S_{-i}} v(s_i, s_{-i}) \prod_{l \neq i} d\sigma_l(s_l) = 0$$

where the last equality again follows from the normalization, $\int_{s_l \in S_l} v(s_l, s_{-l}) d\sigma_l(s_l) = 0$ for all l . Thus, we obtain the desired result. \square

The immediate consequence of Corollary 3.1 and Proposition 5.1 is that every two-player finite normalized zero-sum has a unique uniform mixed strategy Nash equilibrium if Condition (N) holds. If the numbers of two player's strategies are different, all the normalized zero-sum games and normalized common interest games always have a continuum of Nash equilibria containing a uniform mixed strategy, violating Condition (N). The reason is as follows. When the player with a smaller number of strategies plays the uniform mixed strategy, the other player with a larger number of strategies has many mixed strategies giving the same expected payoff to any of the first player's pure strategies. Thus, all these mixed strategies constitute a

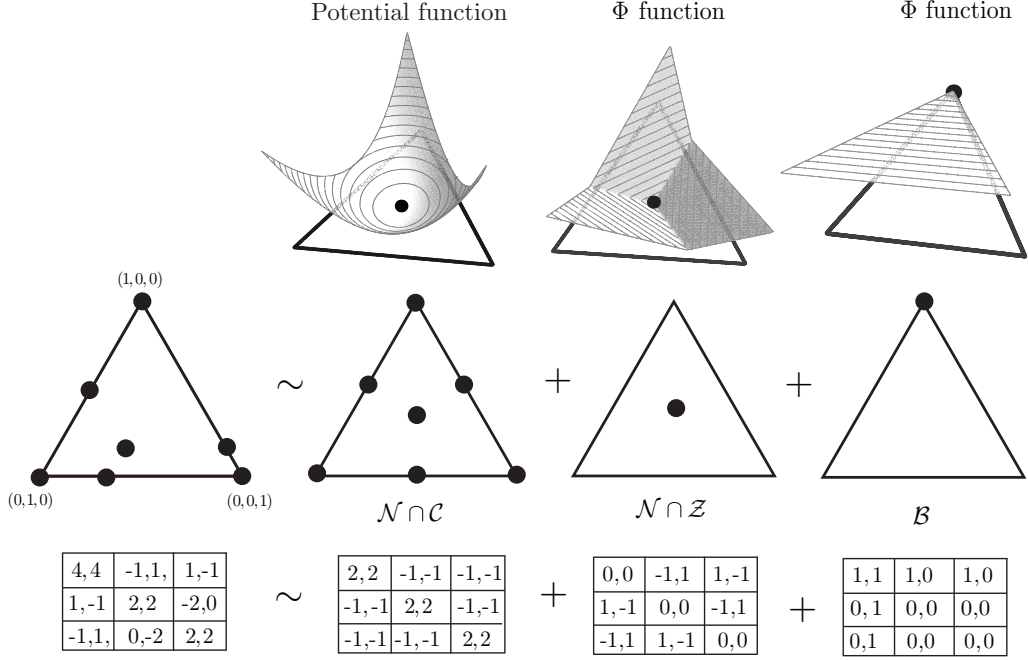


Figure 3: Decomposition of a game into components with distinctive Nash equilibria. In the bottom line we show the two-player game in Table 1. Since all these games are symmetric, we find all symmetric Nash equilibria for the original games (three pure strategy Nash equilibria, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, three mixed strategy Nash equilibria involving two strategies $(1/2, 1/2, 0)$, $(1/6, 0, 5/6)$, $(0, 2/3, 1/3)$, and a completely mixed strategy Nash equilibrium $(1/6, 1/2, 1/3)$) and also find all symmetric Nash equilibria for all the other component games. We show these Nash equilibria using the simplex in the middle line. The top line shows the potential function and the function Φ . The potential function for the zero-sum equivalent potential game in \mathcal{B} is given by p_1 , while the function Φ is given $1 - p_1$, where $p = (p_1, p_2, p_3) \in \Delta$.

Nash equilibrium, yielding a continuum of Nash equilibria (see a similar discussion for harmonic games in Candogan et al. (2011)).

Putting all these ingredients together, we obtain the following decomposition of a given game into components with pure strategy Nash equilibria with a unique uniform mixed Nash equilibrium and a dominant Nash equilibrium (see Figure 3 for an illustration of Theorem 5.1).

Theorem 5.1 (Two-player finite strategy games; Nash equilibria). *Suppose that f is a two-player finite strategy game. Then, f can be uniquely decomposed into three*

components:

$$f \sim f_{NC} + f_{NZ} + f_D$$

where f_{NC} is a normalized common interest game, f_{NZ} is a normalized zero-sum game, and f_D is a zero-sum equivalent potential game. Suppose that all three component games satisfy Condition [\(N\)](#). Then, f_{NC} has a finite number of Nash equilibria with a uniform mixed strategy and f_{NZ} has a unique uniform mixed strategy Nash equilibrium, and f_D a the strictly dominant strategy Nash equilibrium.

Proof. This follows from the decomposition theorem, Theorem [2.1](#), Corollary [4.3](#), and Proposition [5.1](#). \square

The bottom line of Figure [3](#) presents again the decomposition of the symmetric game in Table [1](#). In the middle line of Figure [3](#), we show the Nash equilibria of the original game and the component games in the simplexes. In the top line we show the potential function for the normalized common interest game in $\mathcal{N} \cap \mathcal{C}$, the function Φ for the normalized zero-sum game in $\mathcal{N} \cap \mathcal{Z}$, and the function Φ for the zero-sum equivalent potential game in \mathcal{B} . The decomposition of the game illustrates how each of the Nash equilibria of the original game is related to the Nash equilibria of component games. For example, the necessary condition for the existence of the completely mixed strategy Nash equilibrium for the original game is the existence of the normalized zero-sum or common interest games. Similarly, the existence of the pure strategy Nash equilibria, $(0, 1, 0)$ and $(0, 0, 1)$, is due to the existence of the normalized common interest component. Figure [3](#) also illustrates that if the effect of one component is weak (for example, in terms of payoff sizes), then the original game is expected to be devoid of the desirable properties of the weak component.

Do similar characterizations like Theorem [5.1](#) extend to two-player continuous strategy games and n -player games? First, Proposition [5.1](#) holds for continuous strategy games. In addition, because of the mini-max characterization for two-player continuous strategy zero-sum games (Sion's Theorem), we expect that, under some conditions, the Nash equilibrium for a zero-sum equivalent game is unique and thus a normalized zero-sum game also has a unique uniform mixed strategy and a zero-sum equivalent potential game has a strictly dominant strategy (Corollary [4.1](#)). In fact, using Proposition [3.1](#), we can provide sufficient conditions for a corresponding statement for continuous strategy games like Theorem [5.1](#), which we do not address in this paper. However, for n -player games, structures of zero-sum equivalent potential

games are complicated as shown in Proposition 4.1. Thus, we do not have a good answer yet for n -player games and again leave this question to future research.

Finally, since our characterization relies on the linear structure of payoff functions, it would be useful to know when Nash equilibria remain invariant through linear combination. The following lemma shows one sufficient condition in such a direction.

Lemma 5.1. *Suppose that s^* is a Nash equilibrium for f and f' which have same strategy sets. Let $\rho, \rho' > 0$. Then s^* is a Nash equilibrium for $\rho f + \rho' f'$.*

Proof. We have

$$\begin{aligned}
(\rho f + \rho' f')^{(i)}(s_i^*, s_{-i}^*) &= \rho f^{(i)}(s_i^*, s_{-i}^*) + \rho' (f')^{(i)}(s_i^*, s_{-i}^*) \\
&= \rho \max_{t_i \in S_i} f^{(i)}(t_i, s_{-i}^*) + \rho' \max_{t_i \in S_i} (f')^{(i)}(t_i, s_{-i}^*) \\
&= \max_{t_i \in S_i} \rho f^{(i)}(t_i, s_{-i}^*) + \max_{t_i \in S_i} (\rho' f')^{(i)}(t_i, s_{-i}^*) \\
&\geq \max_{t_i \in S_i} (\rho f + \rho' f')^{(i)}(t_i, s_{-i}^*)
\end{aligned}$$

Thus, s^* is a Nash equilibrium. □

6. Existing decomposition results

Our decomposition methods extend two kinds of existing results: (i) Kalai and Kalai (2010), (ii) Hwang and Rey-Bellet (2011); Candogan et al. (2011). First, Kalai and Kalai (2010) decompose normal form games with incomplete information and study the implications for Bayesian mechanism designs. Their decomposition is based on the orthogonal decomposition $\mathcal{L} = \mathcal{C} \oplus \mathcal{Z}$ in equation (16) in Proposition 2.1.

Second, Hwang and Rey-Bellet (2011) similarly provide decomposition results based on the orthogonality between common interest and zero-sum games and between normalized and non-strategic games, mainly focusing on finite games. Candogan et al. (2011) decompose finite strategy games into three components: a potential component, a nonstrategic component, and a harmonic component. When the numbers of strategies are the same for all players, harmonic components are the same as normalized zero-sum games, and their harmonic games, in this case, refer to games that are strategically equivalent to normalized zero-sum games. Also, their potential component is obtained by removing the non-strategic component from the potential part ($\mathcal{C} + \mathcal{E}$) of the games. Note that we can change our definition of normalized

zero-sum games to their definition of harmonic games, with all the decomposition results remaining unchanged. Thus, their three-component decomposition of finite strategy games follows from Proposition 2.2, $(\mathcal{C} + \mathcal{E}) \oplus (\mathcal{N} \cap \mathcal{Z})$ (see the proof of Corollary 6.1 for more detail).

Corollary 6.1. *We have the following decomposition.*

$$\mathcal{L} = \underbrace{((\mathcal{C} + \mathcal{E}) \cap \mathcal{N})}_{\text{Potential Component}} \oplus \underbrace{\mathcal{E}}_{\text{Nonstrategic Component}} \oplus \underbrace{(\mathcal{N} \cap \mathcal{Z})}_{\text{Harmonic Component}}$$

Proof. See Appendix C. □

Note that Corollary 6.1 not only reproduces the result of Candogan et al. (2011), when the number of strategies of the players is the same, but also extends it to the space of games with continuous strategy sets.

Ui (2000) provides the following characterization for potential games:

$$f \text{ is a potential game if and only if } f^{(i)} = \sum_{\substack{M \subset N \\ M \ni i}} \xi_M \text{ for some } \{\xi_M\}_{M \subset N} \text{ for all } i \quad (43)$$

where ξ_M depends only on s_l , with $l \in M$. From our decomposition results, we have $\mathcal{D} \subset \mathcal{C} + \mathcal{E}$ and $\mathcal{E} \subset \mathcal{C} + \mathcal{D}$. In particular, the second inclusion holds because

$$\zeta_i(s_{-i}) = \sum_{l=1}^n \zeta_l(s_{-l}) - \sum_{l \neq i} \zeta_l(s_{-l}).$$

Thus, $\mathcal{D} \subset \mathcal{C} + \mathcal{E}$ implies that $\mathcal{C} + \mathcal{D} + \mathcal{E} \subset \mathcal{C} + \mathcal{E}$ and $\mathcal{E} \subset \mathcal{C} + \mathcal{D}$ implies $\mathcal{C} + \mathcal{D} + \mathcal{E} \subset \mathcal{C} + \mathcal{D}$. From this, we find

$$\mathcal{C} + \mathcal{D} = \mathcal{C} + \mathcal{D} + \mathcal{E} = \mathcal{C} + \mathcal{E} \quad (44)$$

Note that all games in \mathcal{C} and in \mathcal{D} satisfy Ui's condition in (43); hence, games in $\mathcal{C} + \mathcal{D}$ satisfy Ui's condition. Then, equalities in (44) show that the condition in (43) is a necessary condition for potential games. The sufficiency of Ui's condition

is deduced by adding the non-strategic game

$$\left(\sum_{\substack{M \subset N \\ M \not\ni 1}} \xi_M, \sum_{\substack{M \subset N \\ M \not\ni 2}} \xi_M, \dots, \sum_{\substack{M \subset N \\ M \not\ni n}} \xi_M \right)$$

to game f satisfying Ui's condition.

[Sandholm \(2010a\)](#) decomposes n -player finite strategy games into 2^n components using an orthogonal projection. When the set of games consists of symmetric games with l strategies, the orthogonal projection is given by $\Gamma := I - \frac{1}{l}\mathbf{1}\mathbf{1}^T$, where I is the $l \times l$ identity matrix and $\mathbf{1}$ is the column vector consisting of all 1's. Using Γ , we can, for example, write a given symmetric game, A , as

$$A = \underbrace{\Gamma A \Gamma}_{=(\mathcal{N} \cap \mathcal{C}) \oplus (\mathcal{N} \cap \mathcal{Z})} + \underbrace{(I - \Gamma)A\Gamma + \Gamma A(I - \Gamma) + (I - \Gamma)A(I - \Gamma)}_{=\mathcal{B}}. \quad (45)$$

Thus, our decompositions show that $\Gamma A \Gamma$ can be decomposed further into games with different properties—normalized common interest games and normalized zero-sum games—and every game belonging to the second component in (45) is strategically equivalent to both a common interest game and a zero-sum game. [Sandholm \(2010a\)](#) also shows that a two-player game, (A, B) , is potential if and only if $\Gamma A \Gamma = \Gamma B \Gamma$. If $P = (P^{(1)}, P^{(2)})$ is a non-strategic game, it is easy to see that $\Gamma P^{(1)} = O$ and $P^{(2)}\Gamma = O$, where O is a zero matrix. Thus, the necessity of the condition $\Gamma A \Gamma = \Gamma B \Gamma$ for potential games is obtained. Conversely, if $\Gamma A \Gamma = \Gamma B \Gamma$, then game (A, B) does not have a component belonging to $\mathcal{N} \cap \mathcal{Z}$ because $(\Gamma A \Gamma, \Gamma B \Gamma) \in (\mathcal{N} \cap \mathcal{C}) \oplus (\mathcal{N} \cap \mathcal{Z})$. Thus, (A, B) is a potential game.

7. Conclusion

In this study, we developed decomposition methods for classes of games such as zero-sum equivalent games, zero-sum equivalent potential games, normalized zero-sum games and normalized common interest games. Our methods rely on the orthogonality between common interest and zero-sum games and the orthogonality between normalized and non-strategic games. Using these, we obtained the first decomposition of an arbitrary game into two components—a zero-sum equivalent component and a normalized common interest component—and the second decomposition into

three components—a zero-sum equivalent potential component, a normalized zero-sum component, and a normalized common interest component.

Next, for the class of zero-sum game equivalent games, we showed that each game in this class has a special function, Φ , whose minima are the Nash equilibria of the underlying game. Using this function, we showed the convexity of the set of all Nash equilibria (or the uniqueness of the Nash equilibrium) of a zero-sum equivalent game under some plausible conditions. In particular, we showed that almost all two-player finite zero-sum equivalent games have a unique Nash equilibrium. Using decomposition, we provided characterizations for zero-sum equivalent games. We then studied the class of zero-sum equivalent potential games and showed that almost all two-player finite zero-sum equivalent potential games have a unique strictly dominant Nash equilibrium. We also showed that normalized common interest games and normalized zero-sum games have a uniform mixed strategy Nash equilibrium. Putting all this together, we showed the decomposition of a two-player finite game into component games, each with distinctive Nash equilibrium characterizations (see Figure 3).

Appendix

In Appendix A, we recall the basic decomposition results for a Hilbert space. In Appendix B, we explain in more details our decomposition methods and prove the related results. In Appendix C, we put other proofs and in Appendix D we compare the condition for uniqueness in Proposition 3.1 and the condition by Rosen (1965).

A. Decompositions of a Hilbert space

Let \mathcal{L} be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$. First, we recall some elementary facts about subspaces of the Hilbert space, direct sums, orthogonal projections, and orthogonal decompositions.

If \mathcal{A} and \mathcal{B} are subspaces of \mathcal{L} , then the *sum* of \mathcal{A} and \mathcal{B} , $\mathcal{A} + \mathcal{B}$, is defined as

$$\mathcal{A} + \mathcal{B} := \{x + y; x \in \mathcal{A} \text{ and } y \in \mathcal{B}\},$$

which is again a subspace of \mathcal{L} .

A subspace \mathcal{M} is called the *direct sum* of \mathcal{A} and \mathcal{B} , $\mathcal{A} \oplus \mathcal{B}$, if

- (1) $\mathcal{M} = \mathcal{A} + \mathcal{B}$.
- (2) any $z \in \mathcal{M}$ can be uniquely written as the sum $z = x + y$ with $x \in \mathcal{A}$ and $y \in \mathcal{B}$.

It is easy to see that $\mathcal{M} = \mathcal{A} \oplus \mathcal{B}$ if and only if $\mathcal{M} = \mathcal{A} + \mathcal{B}$ and $\mathcal{A} \cap \mathcal{B} = \{0\}$.

In general, given subspaces \mathcal{M} and $\mathcal{A} \subset \mathcal{M}$, there are many choices of \mathcal{B} such that $\mathcal{M} = \mathcal{A} \oplus \mathcal{B}$. However, in a Hilbert space, there is a canonical choice of \mathcal{B} constructed as follows. If \mathcal{A} is a subspace of a Hilbert space, \mathcal{L} , we denote by \mathcal{A}^\perp its orthogonal complement. That is

$$\mathcal{A}^\perp := \{x \in \mathcal{L}; \langle x, y \rangle = 0 \text{ for all } y \in \mathcal{A}\}.$$

Recall that for any subspace \mathcal{A} , \mathcal{A}^\perp is a closed subspace, and we have $\mathcal{A} \subset \mathcal{A}^{\perp\perp}$. Moreover, $\mathcal{A} = \mathcal{A}^{\perp\perp}$ if and only if \mathcal{A} is a closed subspace of \mathcal{L} . A fundamental theorem in the theory of Hilbert spaces is the following:

Proposition A.1. *Let \mathcal{A} be a closed subspace of \mathcal{L} . Then we have $\mathcal{L} = \mathcal{A} \oplus \mathcal{A}^\perp$.*

Proof. See, for example, Kreyszig (1989). □

Related to orthogonal subspaces is the concept of *orthogonal projections*. An orthogonal projection, T , is a linear map, $T : \mathcal{L} \rightarrow \mathcal{L}$, which (1) is bounded, (2) satisfies $T^2 = T$, and (3) is symmetric (i.e., $\langle Tx, y \rangle = \langle x, Ty \rangle$) for all $x, y \in \mathcal{L}$. Given an orthogonal decomposition, $\mathcal{L} = \mathcal{A} \oplus \mathcal{A}^\perp$, we can define an orthogonal projection, $T_{\mathcal{A}}$, as

$$T_{\mathcal{A}}z := x \text{ if } z \text{ has the decomposition } z = x + y \text{ with } x \in \mathcal{A}, y \in \mathcal{A}^\perp.$$

Conversely, if T is an orthogonal projection, then from Proposition A.1, one obtains an orthogonal decomposition

$$\mathcal{L} = \ker T \oplus \text{range}(T).$$

Note, that even if two subspaces \mathcal{A}, \mathcal{B} are closed, $\mathcal{A} + \mathcal{B}$ is not always closed. However, we have the following two lemmas:

Lemma A.1. *If \mathcal{A} and \mathcal{B} are closed orthogonal subspaces, then $\mathcal{A} \oplus \mathcal{B}$ is closed.*

Proof. Consider the orthogonal projections $T_{\mathcal{A}}$ and $T_{\mathcal{B}}$. Since \mathcal{A} and \mathcal{B} are orthogonal, we have $T_{\mathcal{A}}T_{\mathcal{B}} = 0$ and can easily verify that $T_{\mathcal{A}} + T_{\mathcal{B}}$ is an orthogonal projection onto $\mathcal{A} \oplus \mathcal{B}$, which is then closed. \square

Lemma A.2. *If \mathcal{A} is closed and \mathcal{B} is finite dimensional, then $\mathcal{A} + \mathcal{B}$ is closed.*

Proof. Let $\{z_n\}$ be a convergent sequence in $\mathcal{A} + \mathcal{B}$ (i.e., $z_n = x_n + y_n$, with $x_n \in \mathcal{A}$ and $y_n \in \mathcal{B}$). The sequence $\{y_n\}$ is bounded and by the Bolzano–Weierstrass theorem, has a convergent subsequence, y_{n_k} , which converges to some y in \mathcal{B} . Therefore, x_{n_k} is a convergent sequence and, since \mathcal{A} is closed, x_{n_k} converges to $x \in \mathcal{A}$. \square

The next two results are crucial to our decomposition theorem.

Lemma A.3. *Suppose \mathcal{A} and \mathcal{B} are subspaces of \mathcal{L} . Then,*

- (i) $(\mathcal{A} + \mathcal{B})^\perp = \mathcal{A}^\perp \cap \mathcal{B}^\perp$
- (ii) *If \mathcal{A}, \mathcal{B} , and $\mathcal{A}^\perp + \mathcal{B}^\perp$ are closed, then $\mathcal{A}^\perp + \mathcal{B}^\perp = (\mathcal{A} \cap \mathcal{B})^\perp$.*

Proof. Let $x \in (\mathcal{A} + \mathcal{B})^\perp$. Then, $\langle x, y \rangle = 0$, for all $y \in \mathcal{A} + \mathcal{B}$. Since $\mathcal{A}, \mathcal{B} \subset \mathcal{A} + \mathcal{B}$, we have $\langle x, y_1 \rangle = 0$, for all $y_1 \in \mathcal{A}$, and $\langle x, y_2 \rangle = 0$, for all $y_2 \in \mathcal{B}$. Thus, $(\mathcal{A} + \mathcal{B})^\perp \subset \mathcal{A}^\perp \cap \mathcal{B}^\perp$. Conversely, let $x \in \mathcal{A}^\perp \cap \mathcal{B}^\perp$ and let $y \in \mathcal{A} + \mathcal{B}$. Then, there exist y_1 and y_2 such that $y_1 \in \mathcal{A}$, $y_2 \in \mathcal{B}$, and $y = y_1 + y_2$. Thus, $\langle x, y \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle = 0$.

Therefore, we have $\mathcal{A}^\perp \cap \mathcal{B}^\perp \subset (\mathcal{A} + \mathcal{B})^\perp$. This proves (i). Then, (ii) is a consequence of (i): by (i) and since \mathcal{A} and \mathcal{B} are closed, we have $(\mathcal{A}^\perp + \mathcal{B}^\perp)^\perp = \mathcal{A}^{\perp\perp} \cap \mathcal{B}^{\perp\perp} = \mathcal{A} \cap \mathcal{B}$. Since $\mathcal{A}^\perp + \mathcal{B}^\perp$ is closed, we obtain (ii). \square

Proposition A.2. *Suppose that \mathcal{A} , \mathcal{B} , $\mathcal{A} + \mathcal{B}$, and $\mathcal{A}^\perp + \mathcal{B}$ are all closed subspaces. Then we have the following decompositions:*

- (i) $\mathcal{L} = (\mathcal{A} + \mathcal{B}) \oplus (\mathcal{A}^\perp \cap \mathcal{B}^\perp)$.
- (ii) $\mathcal{L} = (\mathcal{A}^\perp + \mathcal{B}) \oplus (\mathcal{A} \cap \mathcal{B}^\perp)$.
- (iii) $\mathcal{L} = (\mathcal{A} \cap \mathcal{B}^\perp) \oplus (\mathcal{A}^\perp \cap \mathcal{B}^\perp) \oplus ((\mathcal{A} + \mathcal{B}) \cap (\mathcal{A}^\perp + \mathcal{B}))$

Proof. (i) follows from Lemma A.3 (i) and Proposition A.1, since $\mathcal{A} + \mathcal{B}$ is assumed to be closed. (ii) also follows from Lemma A.3 (i) and Proposition A.1, since $\mathcal{A}^\perp + \mathcal{B}$ is closed and \mathcal{A} is closed. For (iii), we first deduce from (i) and (ii) that the subspaces $(\mathcal{A} \cap \mathcal{B}^\perp)$ and $(\mathcal{A}^\perp \cap \mathcal{B}^\perp)$ are closed subspaces. Since they are orthogonal to each other, by Lemma A.1, the direct sum $(\mathcal{A} \cap \mathcal{B}^\perp) \oplus (\mathcal{A}^\perp \cap \mathcal{B}^\perp)$ is also closed. Finally, by applying Lemma A.3 (i) and (ii) again, we obtain

$$((\mathcal{A} \cap \mathcal{B}^\perp) \oplus (\mathcal{A}^\perp \cap \mathcal{B}^\perp))^\perp = (\mathcal{A} \cap \mathcal{B}^\perp)^\perp \cap (\mathcal{A}^\perp \cap \mathcal{B}^\perp)^\perp = (\mathcal{A}^\perp + \mathcal{B}) \cap (\mathcal{A} + \mathcal{B}).$$

Applying Proposition A.1 concludes the proof. \square

B. Decompositions of the space of games

Recall that the space of games is given by the Hilbert space:

$$\mathcal{L} := L^2(S, \mathbb{R}^n; m) = \{f : S \rightarrow \mathbb{R}^n; f \text{ is measurable and } \|f\| < \infty\}.$$

Note that $f \in \mathcal{L}$ if and only if $f^{(i)} \in L^2(S, \mathbb{R}; m) = \{g : S \rightarrow \mathbb{R}; \int |g|^2 dm < \infty\}$, for each $i = 1, 2, \dots, n$. We introduce a number of tools for our decompositions.

Lemma B.1. *Let $T_i : L^2(S, \mathbb{R}, m) \rightarrow L^2(S, \mathbb{R}, m)$ be the map given by*

$$T_i g = \frac{1}{m_i(S_i)} \int g(s) dm_i(s_i).$$

Then, T_i is an orthogonal projection. For $i \neq j$, T_i and T_j commute, and any product $T_{i_1} T_{i_2} \cdots T_{i_k}$ is an orthogonal projection.

Proof. That $T_i^2 = T_i$ follows immediately from the definition, and that T_i is symmetric follows from Fubini's theorem. That T_i and T_j commute again follows from Fubini's theorem, and a product of commuting projections is always a projection. \square

Lemma B.2. *The following maps are orthogonal projections:*

(i) map $\Lambda : \mathcal{L} \rightarrow \mathcal{L}$, defined by

$$\Lambda(f) := (T_1 f^{(1)}, \dots, T_n f^{(n)}).$$

(ii) map $\Phi : \mathcal{L} \rightarrow \mathcal{L}$, defined by

$$\Phi(f) := \left(\frac{1}{n} \sum_i f^{(i)}, \dots, \frac{1}{n} \sum_i f^{(i)} \right).$$

Proof. (i) follows immediately from Lemma B.1, and (ii) follows from an easy computation. \square

Corollary B.1. *The subspaces \mathcal{C} (common interest games), \mathcal{Z} (zero-sum games), \mathcal{E} (non-strategic games), and \mathcal{N} (normalized games) are closed subspaces of \mathcal{L} , and we have*

$$\mathcal{L} = \mathcal{C} \oplus \mathcal{Z} \text{ and } \mathcal{L} = \mathcal{N} \oplus \mathcal{E}.$$

Proof. We have $\mathcal{C} = \text{range}(\Phi)$, $\mathcal{Z} = \ker(\Phi)$, $\mathcal{E} = \text{range}(\Lambda)$, and $\mathcal{N} = \ker(\Lambda)$.

Finally, we need the following key proposition:

Proposition B.1. *We have the following results:*

(i) $\mathcal{C} + \mathcal{E}$ is closed.

(ii) $\mathcal{Z} + \mathcal{E}$ is closed.

Proof. For (i), note first that $\mathcal{C} \cap \mathcal{E}$ is non-empty and contains all constant functions. For this reason, it will be convenient to treat the constant functions separately. Let Υ be given by

$$\Upsilon f := \left(\prod_{l=1}^n T_l f^{(1)}, \prod_{l=1}^n T_l f^{(2)}, \dots, \prod_{l=1}^n T_l f^{(n)} \right).$$

Since Υ and Φ commute, we decompose

$$\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_{\text{const}} \quad \text{and} \quad \mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_{\text{const}},$$

where $\mathcal{C}_{\text{const}} := \text{range}(\Upsilon) \cap \mathcal{C}$, $\mathcal{C}_0 := \ker(\Upsilon) \cap \mathcal{C}$, $\mathcal{E}_{\text{const}} := \text{range}(\Upsilon) \cap \mathcal{E} = \text{range}(\Upsilon)$, and $\mathcal{E}_0 := \ker(\Upsilon) \cap \mathcal{E}$. Since $\mathcal{C}_{\text{const}} + \mathcal{E}_{\text{const}} = \mathcal{E}_{\text{const}}$, we have

$$\mathcal{C} + \mathcal{E} = \mathcal{C}_0 + \mathcal{E}_0 + \mathcal{E}_{\text{const}}.$$

Since $\mathcal{E}_{\text{const}}$ is finite dimensional, by Lemma A.2, it is now enough to show that $\mathcal{C}_0 + \mathcal{E}_0$ is closed.

Let $\{f_k\}_{k=1}^\infty$ be a convergent sequence in $\mathcal{C}_0 + \mathcal{E}_0$, with limit $f = (f^{(1)}, \dots, f^{(2)})$. Then, $f_k = (f_k^{(1)}, f_k^{(2)}, \dots, f_k^{(n)})$ has the form

$$f_k^{(i)} = \phi_k + T_i g_k^{(i)},$$

for all i , and $f_k^{(i)}$ converges to $f^{(i)}$ in $L^2(S, \mathbb{R}; m)$. We next show that, for every i , $\{T_i g_k^{(i)}\}_{k=1}^\infty$ itself is a convergent sequence. Let i be fixed at 1. Since $f \in \mathcal{C}_0 + \mathcal{E}_0$, we have that $T_1 \cdots T_n g_k^{(1)} = 0$, for all k . Then, we write

$$\begin{aligned} T_1 g_k^{(1)} &= T_1(I - T_2)g_k^{(1)} + T_1 T_2 g_k^{(1)} \\ &= T_1(I - T_2)g_k^{(1)} + T_1 T_2(I - T_3)g_k^{(1)} + T_1 T_2 T_3 g_k^{(1)} \\ &= \dots \\ &= \sum_{j=2}^n \left(\prod_{l=1}^{j-1} T_l \right) (I - T_j) g_k^{(1)}, \end{aligned} \tag{B.1}$$

for each k , and we show that each term, $\{(\prod_{l=1}^{j-1} T_l)(I - T_j)g_k^{(1)}\}_k$, in (B.1) converges. Since $\{f_k^{(j)}\}_k$ converges for all j , so do the differences

$$f_k^{(1)} - f_k^{(j)} = T_1 g_k^{(1)} - T_j g_k^{(j)} \longrightarrow f^{(1)} - f^{(j)},$$

for all j . Then, applying $T_1(I - T_2)$ to $f_k^{(1)} - f_k^{(2)}$, we see that $\{T_1(I - T_2)g_k^{(1)}\}_k$ converges. Applying $T_1 T_2(I - T_3)$ to $f_k^{(1)} - f_k^{(3)}$, we see that $\{T_1 T_2(I - T_3)g_k^{(1)}\}_k$ converges and, in general, applying $(\prod_{l=1}^{j-1} T_l)(I - T_j)$ to $f_k^{(1)} - f_k^{(j)}$, we see that $\{(\prod_{l=1}^{j-1} T_l)(I - T_j)g_k^{(1)}\}_k$ converges. Thus, every term in (B.1) converges. Therefore, $\{T_1 g_k^{(1)}\}_k$ converges to some function of the form $T_1 g^{(1)}$, since range T_1 is closed. We can obviously apply the same argument to $T_i g_k^{(i)}$, for all i .

Now, since $\{f_k^{(i)} = \phi_k + T_i g_k^{(i)}\}_k$ converges and $\{T_i g_k^{(i)}\}_k$ converges (in range (T_i)), the sequence $\{\phi_k\}$ must converge to some ϕ . Since \mathcal{C}_0 and \mathcal{E}_0 are closed, we have

$(\phi, \dots, \phi) \in \mathcal{C}_0$ and $(T_1 g^{(1)}, \dots, T_n g^{(n)}) \in \mathcal{E}_0$. Therefore, the limit $f = (f^{(1)}, \dots, f^{(n)})$ belongs to $\mathcal{C}_0 + \mathcal{E}_0$, which shows that $\mathcal{C}_0 + \mathcal{E}_0$ is closed.

For (ii), it will be useful to have the following characterization for $\mathcal{Z} + \mathcal{E}$:

Claim: $f \in \mathcal{Z} + \mathcal{E}$ if and only if $\sum_{i=1}^n f^{(i)} \in \ker(\prod_{l=1}^n (I - T_l))$.

Proof of the claim: If $f \in \mathcal{Z} + \mathcal{E}$, then $f^{(i)} = g^{(i)} + h^{(i)}$, where $\sum_{i=1}^n g^{(i)} = 0$ and $h^{(i)} \in \text{range}(T_i)$. Therefore, we have

$$\sum_{i=1}^n f^{(i)} = \sum_{i=1}^n T_i q^{(i)},$$

for some $q^{(1)}, \dots, q^{(n)}$, and clearly, we have $(\prod_{l=1}^n (I - T_l)) (\sum_{i=1}^n T_i q^{(i)}) = 0$.

Conversely, suppose that $\sum_{i=1}^n f^{(i)} \in \ker(\prod_{l=1}^n (I - T_l))$. Then, we have

$$f^{(i)} = m^{(i)} + n^{(i)},$$

where $n^{(i)} \in \ker(\prod_{l=1}^n (I - T_l))$ and $m^{(i)} \in \text{range}(\prod_{l=1}^n (I - T_l))$, with $\sum_{i=1}^n m^{(i)} = 0$. Since $\ker(\prod_{l=1}^n (I - T_l)) = \text{range}(T_1) + \dots + \text{range}(T_n)$, we have, for each i ,

$$n^{(i)} = \sum_{j=1}^n T_j n_j^{(i)},$$

for some $\{n_j^{(i)}\}_{j=1}^n$. In this way, we find $\{n_j^{(i)}\}_{i,j}$. For each i , we write

$$n^{(i)} = \left(\sum_{j=1}^n T_j n_j^{(i)} - \sum_{j=1}^n T_i n_i^{(j)} \right) + \underbrace{\sum_{j=1}^n T_i n_i^{(j)}}_{\in \text{range } T_i}.$$

Then, we have

$$\sum_{i=1}^n \left(\sum_{j=1}^n T_j n_j^{(i)} - \sum_{j=1}^n T_i n_i^{(j)} \right) = 0.$$

This shows that $f \in \mathcal{Z} + \mathcal{E}$, and concludes the proof of the claim. ■

Now, suppose $\{f_k\}_{k=1}^\infty \in \mathcal{Z} + \mathcal{E}$, such that $f_k \rightarrow f$. Then, $\sum_{i=1}^n f_k^{(i)} \in \ker(\prod_{l=1}^n (I - T_l))$ and $\sum_{i=1}^n f_k^{(i)} \rightarrow \sum_{i=1}^n f^{(i)}$. Since $\ker(\prod_{l=1}^n (I - T_l))$ is closed in $L^2(S, \mathbb{R}; m)$, $\sum_{i=1}^n f^{(i)} \in \ker(\prod_{l=1}^n (I - T_l))$. Thus, $f \in \mathcal{Z} + \mathcal{E}$. □

C. Other proofs

Proof of Proposition 3.2. Let \mathcal{F} be the set of games that satisfy the condition given in (34). First, we verify that $\mathcal{Z} + \mathcal{E} \subset \mathcal{F}$. If $f \in \mathcal{Z}$, then $\sum_i f(s) = 0$, for all s , and thus, (34) holds. If $f \in \mathcal{E}$ and we pick some $i \in N$, then

$$f^{(i)}(s_1, \dots, s_{i-1}, a_i, s_{i+1}, \dots, s_n) - f^{(i)}(s_1, \dots, s_{i-1}, b_i, s_{i+1}, \dots, s_n) = 0$$

and this holds for any s_{-i} . This implies that

$$\sum_{i=1}^n \sum_{s \in \mathcal{S}(a,b)} (-1)^{\#(s)} f^{(i)}(s) = 0. \quad (\text{C.1})$$

Thus, we conclude that $\mathcal{Z} + \mathcal{E} \subset \mathcal{F}$.

To show that $\mathcal{F} \subset \mathcal{Z} + \mathcal{E}$, we show that

$$\mathcal{F} \perp (\mathcal{N} \cap \mathcal{C}).$$

Let $f \in \mathcal{F}$ satisfy condition (34) and let $g \in \mathcal{N} \cap \mathcal{C}$; that is, $g = (v, v, \dots, v)$, with $\int v(s) dm_i(s) = 0$ for every i . Consider t_i as fixed and s_i as varying over S_i . Then, we have, by integrating over s_i ,

$$\int v(s) f^{(i)}(t_i, s_{-i}) dm_i(s_i) = f^{(i)}(t_i, s_{-i}) \int v(s) dm_i(s_i) = 0. \quad (\text{C.2})$$

Now, in condition (34), there are 2^n terms involving $f^{(i)}$ and, unless $s_1 = a_1, s_2 = a_2, \dots, s_n = a_n$, $f^{(i)}$ is evaluated as some $t_j = b_j$. Thus, if we multiply both sides of (34) by $v(s)$ and integrate, we find, by (C.1) and Fubini's theorem,

$$\begin{aligned} 0 &= \int v(s) \sum_{i=1}^n \sum_{s \in \mathcal{S}(a,b)} (-1)^{\#(s)} f^{(i)}(s) dm(s) \\ &= \int v(s) \sum_{i=1}^n f^{(i)}(s_1, s_2, \dots, s_n) dm(s). \end{aligned}$$

However, this simply means that $\langle f, g \rangle = 0$, for $f \in \mathcal{F}$ and $g \in \mathcal{N} \cap \mathcal{C}$. From Theorem 2.1 (i), we find that $(\mathcal{N} \cap \mathcal{C}) \perp \mathcal{F}$ and $\mathcal{F} = \mathcal{Z} + \mathcal{E}$. \square

Proof of Corollary 3.2. From (31), if the game is symmetric, then we obtain the following condition for a zero-sum equivalent game.

$$f^{(1)}(s_1, s_2) + f^{(1)}(s_2, s_1) - f^{(1)}(t_1, s_2) - f^{(1)}(s_2, t_1) - f^{(1)}(s_1, t_2) - f^{(1)}(t_2, s_1) + f^{(1)}(t_1, t_2) + f^{(1)}(t_2, t_1) = 0 \quad (\text{C.3})$$

If we set $s = s_1 = s_2$ and $t = t_1 = t_2$, then equation (C.3) implies condition in (35). Now suppose that condition in (35) holds. Then we have

$$f^{(1)}(s, s) + f^{(1)}(t, t) = f^{(1)}(s, t) + f^{(1)}(t, s)$$

and thus

$$\begin{aligned} & f^{(1)}(s_1, s_2) + f^{(1)}(s_2, s_1) - f^{(1)}(t_1, s_2) - f^{(1)}(s_2, t_1) \\ & - f^{(1)}(s_1, t_2) - f^{(1)}(t_2, s_1) + f^{(1)}(t_1, t_2) + f^{(1)}(t_2, t_1) \\ & = f^{(1)}(s_1, s_1) + f^{(1)}(s_2, s_2) - f^{(1)}(t_1, t_1) - f^{(1)}(s_2, s_2) \\ & - f^{(1)}(s_1, s_1) - f^{(1)}(t_2, t_2) + f^{(1)}(t_1, t_1) + f^{(1)}(t_2, t_2) \\ & = 0 \end{aligned}$$

and thus (C.3) holds. □

Proof of Proposition 3.3. Let i and l be fixed, and suppose that

$$f^{(i)}(s) = \frac{1}{m_l(S_l)} \int f^{(i)}(s_l, s_{-l}) dm_l.$$

If $l = i$, then $f = (0, \dots, \underbrace{f^{(i)}(s)}_{i\text{-th}}, \dots, 0)$, which is a passive game and, hence, is strategically equivalent to a zero-sum game. Thus, suppose that $l \neq i$. Consider a component, $(0, \dots, \underbrace{-f^{(i)}(s)}_{l\text{-th}}, \dots, 0)$. Then, this component is again a passive game.

In this way, for all i , we can define a passive game such that the sum of f and all resulting passive games is zero. Thus, f is strategically equivalent to a zero-sum game. □

Proof of Proposition 3.4. Recall that

$$\pi^{(i)}((s_1, \dots, s_k), (s_{k+1}, \dots, s_{2k})) = \begin{cases} \sum_{l=k+1}^{2k} f^{(1)}(s_i, s_l, \tau_1^{(i)}, \tau_2^{(l)}) p_l q_l & \text{if } i \leq k \\ \sum_{l=1}^k f^{(2)}(s_l, s_i, \tau_1^{(l)}, \tau_2^{(i)}) p_l q_i & \text{if } i \geq k+1 \end{cases}. \quad (\text{C.4})$$

Then from Proposition ..., π is strategically equivalent to a zero-sum game. Suppose that $(f^{(1)}(s_\alpha, s_\beta, \tau_1, \tau_2), f^{(2)}(s_\alpha, s_\beta, \tau_1, \tau_2))$ is a potential game, for each τ_1 and τ_2 . Let i and l be fixed. Without loss of generality, $i \leq k$ and $l \geq k+1$. Consider

$$(0, \dots, \underbrace{f^{(1)}(s_i, s_l, \tau_1^{(i)}, \tau_2^{(l)}) p_l q_l}_{i\text{-th}}, 0, \dots, 0, \underbrace{f^{(2)}(s_l, s_i, \tau_1^{(l)}, \tau_2^{(i)}) p_l q_i}_{j\text{-th}}, 0, \dots, 0). \quad (\text{C.5})$$

Since $(f^{(1)}, f^{(2)})$ is a potential game, $(f^{(1)}, f^{(2)}) \sim (g, g)$, for some g . Then,

$$(g(s_i, s_l), g(s_i, s_l), \dots, g(s_i, s_l), \underbrace{0}_{i\text{-th}}, g(s_i, s_l), \dots, g(s_i, s_l), \underbrace{0}_{j\text{-th}}, g(s_i, s_l), \dots)$$

is a passive game. Thus, the component in (C.5) is a potential game. Similarly, we can show that other components are potential games, which shows that π is a potential game. \square

Proof of Proposition 4.1. We first show that

$$\mathcal{B} = (\mathcal{Z} + \mathcal{E}) \cap (\mathcal{C} + \mathcal{E}) = \Phi(\Lambda(\mathcal{L})) + \mathcal{E}.$$

Let $f \in (\mathcal{Z} + \mathcal{E}) \cap (\mathcal{C} + \mathcal{E})$. Then, $f = g_1 + h_1$, for $g_1 \in \mathcal{Z}$ and $h_1 \in \mathcal{E}$, and $f = g_2 + h_2$, for $g_2 \in \mathcal{C}$ and $h_2 \in \mathcal{E}$. Thus, we have

$$g_1 + h_1 = g_2 + h_2, \quad (\text{C.6})$$

and applying Φ to (C.6), we obtain

$$f = \Phi(h_1 - h_2) + h_2.$$

Thus, since $h_1 - h_2 \in \Lambda(\mathcal{L})$, $f \in \Phi(\Lambda(\mathcal{L})) + \mathcal{E}$. Conversely, let $f \in \Phi(\Lambda(\mathcal{L})) + \mathcal{E}$.

Obviously, $f \in \mathcal{C} + \mathcal{E}$. In addition, $f = \Phi(\Lambda(g)) + h_1$, for $g \in \mathcal{L}$ and $h_1 \in \mathcal{E}$. Thus,

$$f = \Phi(\Lambda(g)) + h_1 = -(I - \Phi)(\Lambda(g)) + \Lambda(g) + h_1 \in \mathcal{Z} + \mathcal{E}.$$

This shows that

$$(\mathcal{Z} + \mathcal{E}) \cap (\mathcal{C} + \mathcal{E}) = \Phi(\Lambda(g)) + \mathcal{E}.$$

Note that

$$\begin{aligned} \Phi(\Lambda(\mathcal{L})) + \mathcal{E} &= \{f : f^{(i)} = \sum_{l=1}^n \zeta_l(s_{-l}) \text{ for some } \{\zeta_l\}_{l=1}^n \text{ and for all } i\} + \mathcal{E} \\ &= \{f : f^{(i)} = \sum_{l \neq i}^n \zeta_l(s_{-l}) \text{ for some } \{\zeta_l\}_{l=1}^n \text{ and for all } i\} + \mathcal{E} \\ &= \mathcal{D} + \mathcal{E}. \end{aligned}$$

□

Proof of Proposition 4.2. Observe that

$$\begin{aligned} &(\sum_{l \neq 1} g_l(s_{-l}), \sum_{l \neq 2} g_l(s_{-l}), \dots, \sum_{l \neq n} g_l(s_{-l})) \\ &\sim (\sum_{l=1}^n g_l(s_{-l}), \sum_{l=2}^n g_l(s_{-l}), \dots, \sum_{l=1}^n g_l(s_{-l})). \end{aligned}$$

Hence, the first result follows from $\mathcal{D} + \mathcal{E} = (\mathcal{C} + \mathcal{E}) \cap (\mathcal{Z} + \mathcal{E})$. For the second result,

observe that

$$\begin{aligned}
(\sum_{l \neq 1} g_l, \sum_{l \neq 2} g_l, \dots, \sum_{l \neq n} g_l) &\sim (\sum_{l \neq 1} g_l - (n-1)g_1, \sum_{l \neq 2} g_l - (n-1)g_2, \dots, \sum_{l \neq n} g_l - (n-1)g_n) \\
&= (\sum_{l \neq 1} (g_l - g_1), \sum_{l \neq 2} (g_l - g_2), \dots, \sum_{l \neq n} (g_l - g_n)) \\
&= (\sum_{l > 1} (g_l - g_1), g_1 - g_2, g_1 - g_3, \dots, g_1 - g_n) \\
&\quad + (0, \sum_{l > 2} (g_l - g_2), g_2 - g_3, \dots, g_2 - g_n) + \dots \\
&\quad + (0, 0, \dots, \sum_{l > n-1} (g_l - g_{n-1}), g_{n-1} - g_n) \\
&= \sum_{i=1}^n \sum_{l > i}^n (0, \dots, 0, \underbrace{-g_i + g_j}_{i\text{-th}}, 0, \dots, 0, \underbrace{g_i - g_j}_{j\text{-th}}, 0, \dots, 0).
\end{aligned}$$

□

Proof of Corollary 6.1. This proof follows from Proposition 2.2 by showing that $((\mathcal{C} + \mathcal{E}) \cap \mathcal{N}) \oplus \mathcal{E} = \mathcal{C} + \mathcal{E}$. First, observe that $(\mathcal{C} + \mathcal{E}) \cap \mathcal{N} \subset \mathcal{C} + \mathcal{E}$, which implies that $((\mathcal{C} + \mathcal{E}) \cap \mathcal{N}) \oplus \mathcal{E} \subset \mathcal{C} + \mathcal{E}$. Now, let $f \in \mathcal{C} + \mathcal{E}$. Then, $f = g + h$, where $g \in \mathcal{C}$, $h \in \mathcal{E}$, and $g = (v, v, \dots, v)$. Then, by applying the map, Λ , we find that $f = \Lambda(f) + (I - \Lambda)(f)$. Obviously, $\Lambda(f) \in \mathcal{E}$. In addition, $(I - \Lambda)(f) = (I - \Lambda)(g) = (v - T_1 v, v - T_2 v, \dots, v - T_n v) \in \mathcal{C} + \mathcal{E}$. Thus, $(I - \Lambda)(f) \in (\mathcal{C} + \mathcal{E}) \cap \mathcal{N}$. □

D. Comparison of the condition for uniqueness in Proposition 3.1 and the condition by Rosen (1965)

To simplify we consider the two player zero-sum equivalent game given by

$$\begin{aligned}
f^{(1)}(s_1, s_2) &= w^{(1)}(s_1, s_2) + h^{(1)}(s_2) \\
f^{(2)}(s_1, s_2) &= w^{(2)}(s_1, s_2) + h^{(2)}(s_1)
\end{aligned}$$

and suppose that $w^{(1)}$ and $w^{(2)}$ are differentiable. Then Proposition 3.1 (ii) requires that

$$\frac{\partial^2 w^{(1)}}{\partial s_1^2} < 0 \text{ and } \frac{\partial^2 w^{(1)}}{\partial s_2^2} > 0.$$

Also, the sufficient condition for the uniqueness of the Nash equilibrium for concave games by Rosen (1965) (Theorems 2 and 6) is given by

$$\begin{pmatrix} 2r_1 \frac{\partial^2 w^{(1)}}{\partial s_1^2} & r_1 \frac{\partial^2 w^{(1)}}{\partial s_1 \partial s_2} + r_2 \frac{\partial^2 w^{(2)}}{\partial s_1 \partial s_2} \\ r_1 \frac{\partial^2 w^{(1)}}{\partial s_1 \partial s_2} + r_2 \frac{\partial^2 w^{(2)}}{\partial s_1 \partial s_2} & 2r_2 \frac{\partial^2 w^{(2)}}{\partial s_2^2} \end{pmatrix} = \begin{pmatrix} 2r_1 \frac{\partial^2 w^{(1)}}{\partial s_1^2} & r_1 \frac{\partial^2 w^{(1)}}{\partial s_1 \partial s_2} - r_2 \frac{\partial^2 w^{(1)}}{\partial s_1 \partial s_2} \\ r_1 \frac{\partial^2 w^{(1)}}{\partial s_1 \partial s_2} - r_2 \frac{\partial^2 w^{(1)}}{\partial s_1 \partial s_2} & -2r_2 \frac{\partial^2 w^{(1)}}{\partial s_2^2} \end{pmatrix}$$

is negative definite for some $r_1, r_2 > 0$. Thus Rosen's condition requires some restriction on the cross partial derivatives, while the condition in Proposition 3.1 does not.

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